Strict Implication Over Logics Without Contraction

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Strict implication and substructural implication

- Strict implication arises from prefixing material implication by a modal necessity operator: $\Box(\varphi \to \psi)$.
- Important in Lewis's work on early modern modal logic for resolving paradoxes of material implication.
- Modal logics above S4 have an especially nice strict implication, corresponding (via Gödel's translation) to intuitionistic implication.
- Substructural logics generalize intuitionistic logic.
- **Guiding question:** Which substructural logics are logics of strict implication (over some non-classically based modal logic)?

Logics without contraction

- Substructural logics dispense with some of the structural rules arising in the proof theory of classical/intuitionistic logic.
- Today: Logics with exchange and weakening, but missing the contraction rule of the intuitionistic sequent calculus LJ:

$$\frac{\Gamma, \varphi, \varphi, \Rightarrow \Sigma}{\Gamma, \varphi, \Rightarrow \Sigma}$$

Algebraic semantics:

A (bounded, commutative, integral) residuated lattice is an algebra $(A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ such that

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.
- $(A, \cdot, 1)$ is a commutative monoid. ($\cdot$ interprets comma)
- For all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq y \rightarrow z.$$

Familiar residuated lattices

Lots of familiar examples:

- Heyting algebras (where $\cdot$ is $\wedge$) and Boolean algebras.
- MTL-algebras, the algebraic semantics of t-norm based logics, satisfying $(x \to y) \vee (y \to x) = 1$ (residuated lattices that are subdirect products of totally ordered ones).
- GBL-algebras, satisfying divisibility $x(x \to y) = x \wedge y$.
- BL-algebras, the algebraic semantics of Petr Hájek's basic fuzzy logic, the intersection of MTL and GBL.
- MV-algebras, the algebraic semantics of Łukasiewicz logic, BL-algebras that satisfy $(x \to 0) \to 0 = x$.
- Gödel algebras, the algebraic semantics of Gödel-Dummett logic, the intersection of MTL and Heyting algebras.

Non-classical frames

Definition:

A frame is an ordered triple $(X, \leq, \mathbb{A})$, where

- $(X, \leq)$ is a poset.
- $\mathbb{A} = \{\mathbf{A}_x : x \in A\}$ is an indexed family of residuated lattices.

If K is a class of posets, we say that the frame $(X, \leq, \mathbb{A})$ is K-based or based in K when $(X, \leq) \in$ K. Likewise, if V is a class of residuated lattices, we say that $(X, \leq, \mathbb{A})$ is V-valued or valued in V when $\mathbf{A}_x \in$ V for every $x \in X$.

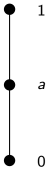
Valuations in intuitionistic frames satisfy persistency: If φ is true at a world x, then φ remains true at each $y \geq x$. The correct generalization to non-classical frames is the notion of an antichain labeling.

Antichain labelings

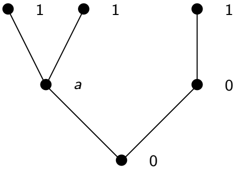
Definition:

Let $(X, \leq)$ be a poset, and let $\{\mathbf{A}_x : x \in X\}$ be an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. An antichain labeling (or ac-labeling) is a choice function $f \in \prod_{x \in X} A_x$ such that for all $x, y \in X$,

$$x < y \implies f(x) = 0 \text{ or } f(y) = 1.$$



$\mathbf{A}_x$



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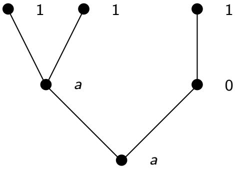

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Antichain labelings as $\Box$-fixed elements

Let $(X, \leq)$ be a poset and $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. Set $\mathbf{B} = \prod_{x \in X} \mathbf{A}_x$ and define a map $\Box \colon B \to B$ by

$$\Box(f)(x) = \begin{cases} f(x) & \text{if } f(y) = 1 \text{ for all } y > x \\ 0 & \text{if there exists } y > x \text{ with } f(y) \neq 1. \end{cases}$$

Then:

- $\Box$ is an interior operator on $\mathbf{B}$.
- In particular, $\Box\Box x = \Box x \leq x$ and $\Box$ preserves $\wedge$ and $\cdot$, and $\Box(x \to y) \to (\Box x \to \Box y) = 1$.
- The $\Box$-fixed elements are exactly the ac-labelings.
- The image of $\mathbf{B}$ under $\Box$ is a residuated lattice $\mathbf{B}_\Box$; all the operations are from $\mathbf{B}$, except the new implication is $\Box(x \to y)$, where $\to$ is the implication of $\mathbf{B}$. The algebra $\mathbf{B}_\Box$ is called the poset product.

Thinking about poset products

Poset products were originally introduced by P. Jipsen and. F. Montagna as a common generalization of direct products and nested sums (sometimes called ordinal sums).

- If $(X, =)$ is the index poset, then the poset product of $\{\mathbf{A}_x : x \in X\}$ is just the direct product.
- If $x < y$ in the poset $(\{x, y\}, \leq)$, then the poset product consists of the nested sum of $\mathbf{A}_x$ and $\mathbf{A}_y$ (intuitively obtained by replacing the unit of $\mathbf{A}_x$ by $\mathbf{A}_y$).

Poset products can be thought of as iterating the direct product and nested sum constructions.

Toward algebraic models of strict implication

Poset product representations realize residuated lattices as models of substructural logics of strict implication.

- The representations are most useful when the factors $\mathbf{A}_x$ have much lower complexity than the algebras of interest.
- We focus on the case with simple factors: Where the only congruences are the trivial ones.

Theorem (Kowalski-Ono, 2000):

Let $\mathbf{A}$ be a simple residuated lattice and let $a \in A$ with $a \neq 1$. Then there exists $n \in \mathbb{N}$ such that $a^n = 0$.

- Simple residuated lattices are in particular multipotent: For each a there exists $n \in \mathbb{N}$ such that $a^{n+1} = a^n$.
- This highlights the role idempotents play in poset products of simple residuated lattices, i.e. since simple ones have no idempotents other than $0, 1$.

Some definitions

Definition (idempotent center):

- The idempotent center of the residuated lattice $\mathbf{A}$ is the set $\mathcal{H}(A) = \{a \in A : a^2 = a\}$.
- If $\mathcal{H}(A)$ is a (necessarily Heyting) subalgebra of $\mathbf{A}$ and for all $i \in \mathcal{H}(A)$, $a \in A$ we have $ia = i \wedge a$, we say that it is a central subalgebra of $\mathbf{A}$ and denote it by $\mathcal{H}(\mathbf{A})$.

Definition (central filters):

- A filter of a residuated lattice $\mathbf{A}$ is a subset that is upward closed and closed under $\cdot$.
- For each subset S of A, there is a smallest filter containing S called the filter generated by S.
- A filter is called central if it is the filter generated the idempotent elements it contains.
- A value is completely meet irreducible element in the lattice of filters.

Centered residuated lattices

Representability by poset products of simple residuated lattices turns out to depend crucially on $\mathcal{H}(A)$ fitting inside **A** 'nicely':

Definition:

We say that a residuated lattice **A** is centered if:

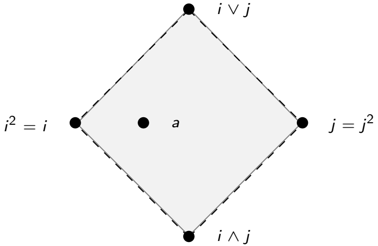
- $\mathcal{H}(\mathbf{A})$ is a central subalgebra of **A**.
- Every filter of **A** is central.
- **A** satisfies the square condition: For every $i \in \mathcal{H}(A)$ and $a \in A$, there exists $j \in \mathcal{H}(A)$ such that $i \wedge j \leq a \leq i \vee j$.

Theorem (F.-Jipsen 2022+):

Every centered residuated lattice embeds into a poset product of simple residuated lattices, and is therefore isomorphic to an algebra of antichain labelings.

The square condition

The square condition: For every $i \in \mathcal{H}(A)$ and $a \in A$, there exists $j \in \mathcal{H}(A)$ such that $i \wedge j \leq a \leq i \vee j$.



Sketch of the main theorem

- Let $(X, \leq)$ be the poset of completely meet-irreducible congruences of the centered residuated lattice $\mathbf{A}$.
- The quotients $\mathbf{A}/x$ for $x \in X$ are subdirectly irreducible.
- Key lemma: These may be decomposed as a nested sum with one summand a simple residuated lattices (a Blok-Ferreirim theorem for centered residuated lattices).
- $\mathbf{A}$ may be embedded in the poset product of these factors indexed by the poset of completely meet-irreducible congruences.

Varieties and logics

Centered residuated lattices don't form an especially nice class, and what we're interested in for logical purposes are varieties.

Definition:

For each $n \in \mathbb{N}$, let S_n denote the subvariety of residuated lattices axiomatized by:

- $a^n b = a^n \wedge b$.
- $a^n \to b^n = (a^n \to b^n)^2$.
- $a \leq b^n \vee (b^n \to a^n)$.

Further, for each $n \in \mathbb{N}$ denote by C_n the subvariety of S_n axiomatized by

$$(a \to b) \to (b \to a) = b \to a.$$

Concrete representation theorems

Theorem (Jipsen-Montagna 2010):

For each $n \in \mathbb{N}$, the variety generated by poset products of simple n-potent MV-algebras chains is the variety of n-potent GBL-algebras.

Theorem (F.-Jipsen 2022+):

Let $n \in \mathbb{N}$.

- S_n is the variety generated by poset products of simple n-potent residuated lattices.
- C_n is the variety generated by poset products of simple n-potent MTL-algebras.

Conclusion

- This framework gives lots of substructural analogues of notions from classical modal/intuitionistic logic:
  - Relational semantics
  - Sahlqvist theory
  - Gödel-McKinsey-Tarski type translations
- On-going and future work:
  - Add topological content to what we've seen, extending Esakia duality to the substructural setting.
  - Go beyond simple factors for more expressive representation theories.
  - Develop a substructural Blok-Esakia theory of modal companions.

Thank you!

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