Representations by Antichain Labelings

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Part I:
Antichain labelings and poset products

Intuitionistic Kripke semantics

An intuitionistic Kripke frame is a just a poset $(X, \leq)$.

- We think of the elements of X as 'possible worlds' or more prosaically 'situations'.
- The truth/falsity of each proposition φ is evaluated at each world $x \in X$.
- Connectives $\wedge, \vee$ are evaluated locally at each world, but instead of the material implication $\supset$ we take the strict implication: $\varphi \to \psi$ is true at $x \in X$ if for each $y \geq x$, $\varphi \supset \psi$ holds at y.
- Persistency is a key condition: If φ is true at $x \in X$, then φ remains true at each $y \geq x$.
- Said differently: The set of points $x \in X$ at which φ is true is an up-set of $(X, \leq)$.

Up-sets and Heyting algebras

- The up-sets of any poset satisfies the frame law, and hence forms a complete Heyting algebra.
- In fact, every Heyting algebra embeds in the algebra of up-sets of its poset of prime filters (sometimes called its canonical extension/completion).
- Different perspective: Swap out each up-set U for its characteristic functions $\chi_U \colon X \to \{0,1\}$ defined by $\chi_U(x) = 1$ iff $x \in U$. These are just monotone functions from $(X, \leq)$ to the 2-element Boolean algebra $\{0,1\}$.
- Core idea of antichain labelings.
- In-line with representation theory throughout algebra (groups of permutations, matrices, Boolean algebras, etc).

Generalizing persistency

- Motivated by many-valued logics: Might want worlds to values not just in $\{0, 1\}$, but with other intermediate truth values (example: MV-algebras and Łukasiewicz logic).
- Lots of approaches in the literature, not yet clear what the 'correct' way to do this is.
- Monotonicity is no longer enough.
- Logic and representation theory guide each other: Considering generalizations of frames leads us to more expressive representation theories and representation theory helps select the 'correct' generalization of persistency/monotonicity.

Residuated lattices

Definition:

A (bounded, commutative, integral) residuated lattice is an algebra $(A, \wedge, \vee, \cdot, \to, 0, 1)$ such that

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.
- $(A, \cdot, 1)$ is a commutative monoid.
- For all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq y \to z.$$

We will usually write xy for $x \cdot y$. Residuated lattices give the equivalent algebraic semantics for extensions of the Full Lambek calculus (with exchange, weakening, and falsum). We drop contraction from intuitionistic sequent calculus LJ. The product $\cdot$ interprets comma.

Familiar residuated lattices

Lots of familiar examples:

- Heyting algebras (where $\cdot$ is $\wedge$) and Boolean algebras.
- MTL-algebras, the algebraic semantics of t-norm based logics, satisfying $(x \to y) \vee (y \to x) = 1$ (residuated lattices that are subdirect products of totally ordered ones).
- GBL-algebras, satisfying divisibility $x(x \to y) = x \wedge y$.
- BL-algebras, the algebraic semantics of Petr Hájek's basic fuzzy logic, the intersection of MTL and GBL.
- MV-algebras, the algebraic semantics of Łukasiewicz logic, BL-algebras that satisfy $(x \to 0) \to 0 = x$.
- Gödel algebras, the algebraic semantics of Gödel-Dummett logic, the intersection of MTL and Heyting algebras.

Non-classical frames

Most of the results will not be phrased in this language, but we can make precise the idea of non-classical 'frames' we discussed before:

Definition:

A frame is an ordered triple $(X, \leq, \mathbb{A})$, where

- $(X, \leq)$ is a poset.
- $\mathbb{A} = \{\mathbf{A}_x : x \in A\}$ is an indexed family of residuated lattices.

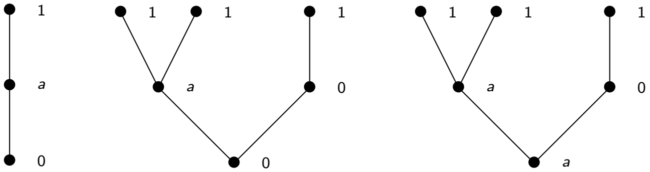
If K is a class of posets, we say that the frame $(X, \leq, \mathbb{A})$ is K-based or based in K when $(X, \leq) \in \mathsf{K}$. Likewise, if V is a class of residuated lattices, we say that $(X, \leq, \mathbb{A})$ is V-valued or valued in V when $\mathbf{A}_x \in \mathsf{V}$ for every $x \in X$.

Antichain labelings

Definition:

Let $(X, \leq)$ be a poset, and let $\{\mathbf{A}_x : x \in X\}$ be an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. An antichain labeling (or ac-labeling) is a choice function $f \in \prod_{x \in X} A_x$ such that for all $x, y \in X$,

$$x < y \implies f(x) = 0 \text{ or } f(y) = 1.$$



$\mathbf{A}_x$ Good Bad

Poset products

Definition:

Let $(X, \leq)$ be a poset and let $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and greatest element 1. Set $B = \{f \in \prod_{x \in X} : f$ is an ac-labeling$\}$. We define operations in B as follows. The operations $\wedge, \vee, \cdot, 0, 1$ are defined pointwise, and the operation $\rightarrow$ is defined by

$$(f \rightarrow g)(x) = \begin{cases} f(x) \rightarrow_x g(x) & \text{if for all } y > x, f(y) \leq_x g(y) \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $\mathbf{B}$ with these operation is called the poset product.

Fact: A poset product of a poset-indexed family of residuated lattices is a residuated lattice.

If **A** is a residuated lattice, a map $\sigma\colon A \to A$ is a conucleus on **A** if for all $x, y \in A$:

- $\sigma(x) \leq x$
- $\sigma(\sigma(x)) = \sigma(x)$.
- $x \leq y$ implies $\sigma(x) \leq \sigma(y)$
- $\sigma(x)\sigma(y) \leq \sigma(xy)$
- $\sigma(x)\sigma(1) = \sigma(1)\sigma(x) = \sigma(x)$

If σ is a conucleus on $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 0, 1)$, then

$$\mathbf{A}_\sigma = (\sigma[A], \wedge_\sigma, \vee, \cdot, \to_\sigma, 0, \sigma(1))$$

is also a residuated lattice, where $x \wedge_\sigma y = \sigma(x \wedge y)$ and $x \to_\sigma y = \sigma(x \to y)$.

Poset products as conuclear images

Let $(X, \leq)$ be a poset and $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. Set $\mathbf{B} = \prod_{x \in X} \mathbf{A}_x$ and define a map $\Box \colon B \to B$ by

$$\Box(f)(x) = \begin{cases} f(x) & \text{if } f(y) = 1 \text{ for all } y > x \\ 0 & \text{if there exists } y > x \text{ with } f(y) \neq 1. \end{cases}$$

Then $\Box$ is a conucleus on the direct product. The conuclear image coincides with the poset product:

$$\mathbf{B}_{\Box} = \prod_{(X,\leq)} \mathbf{A}_x.$$

Describing ac-labelings

Antichain labelings admit many convenient equivalent characterizations:

Lemma:

Let $f \in \mathbf{B} = \prod_{x \in X} \mathbf{A}_x$, $(X, \leq)$ a poset, as above. The following are equivalent.

1. $f \in \mathbf{B}_\Box$.
2. $\Box f = f$.
3. For all $x, y \in X$ with $x < y$, $f(x) = 0$ or $f(y) = 1$.
4. $S_f = \{x \in X : f(x) \notin \{0, 1\}\}$ is a (possibly empty) antichain of $(X, \leq)$, $L_f = f^{-1}(0)$ is a down-set of $(X, \leq)$, and $U_f = f^{-1}(1)$ is an up-set of $(X, \leq)$.

Thinking about poset products

Poset products were originally introduced by P. Jipsen and. F. Montagna as a common generalization of direct products and nested sums (sometimes called ordinal sums).

- If $(X, =)$ is the index poset, then the poset product of $\{\mathbf{A}_x : x \in X\}$ is just the direct product.
- If $x < y$ in the poset $(\{x, y\}, \leq)$, then the poset product consists of the nested sum of $\mathbf{A}_x$ and $\mathbf{A}_y$ (intuitively obtained by replacing the unit of $\mathbf{A}_x$ by $\mathbf{A}_y$).

Poset products can be thought of as iterating the direct product and nested sum constructions.

Poset product representations

Recall that a GBL-algebra is a residuated lattice that satisfies divisibility ($x(x \to y) = x \wedge y$). Almost all of the past work on poset product representations has been directed at GBL-algebras and BL-algebras (the subvariety generated by totally ordered GBL-algebras).

Theorem (Jipsen-Montagna 2010):

- Every GBL-algebra embeds in a poset product of totally ordered MV-algebras.
- Every n-potent GBL-algebra (satisfying $x^{n+1} = x^n$) embeds into a poset product of finite simple n-potent MV-algebra chains.

Consequences: Decidability for universal theory of GBL (Jipsen-Montagna), amalgamation for some subvarieties (Metcalfe-Montagna-Tsinakis), etc.

Part II:
Representations beyond divisibility

New representations

We'll head toward some poset product representations for non-divisible residuated lattices.

- The representations are most useful when the factors $\mathbf{A}_x$ have much lower complexity than the algebras of interest.
- We focus on the case with simple factors: Where the only congruences are the trivial ones.

Theorem (Kowalski-Ono, 2000):

Let $\mathbf{A}$ be a simple residuated lattice and let $a \in A$ with $a \neq 1$. Then there exists $n \in \mathbb{N}$ such that $a^n = 0$.

- Simple residuated lattices are in particular multipotent: For each a there exists $n \in \mathbb{N}$ such that $a^{n+1} = a^n$.
- This highlights the role idempotents play in poset products of simple residuated lattices, i.e. since simple ones have no idempotents other than $0, 1$.

Some definitions

Definition (idempotent center):

- The idempotent center of the residuated lattice **A** is the set $\mathcal{H}(A) = \{a \in A : a^2 = a\}$.
- If $\mathcal{H}(A)$ is a (necessarily Heyting) subalgebra of **A** and for all $i \in \mathcal{H}(A)$, $a \in A$ we have $ia = i \wedge a$, we say that it is a central subalgebra of **A** and denote it by $\mathcal{H}(\mathbf{A})$.

Definition (central filters):

- A filter of a residuated lattice **A** is a subset that is upward closed and closed under $\cdot$.
- For each subset S of A, there is a smallest filter containing S called the filter generated by S.
- A filter is called central if it is the filter generated the idempotent elements it contains.
- A value is completely meet irreducible element in the lattice of filters.

Centered residuated lattices

Representability by poset products of simple residuated lattices turns out to depend crucially on $\mathcal{H}(A)$ fitting inside $\mathbf{A}$ 'nicely':

Definition:

We say that a residuated lattice $\mathbf{A}$ is centered if:

- $\mathcal{H}(\mathbf{A})$ is a central subalgebra of $\mathbf{A}$.
- Every filter of $\mathbf{A}$ is central.
- $\mathbf{A}$ satisfies the square condition: For every $i \in \mathcal{H}(A)$ and $a \in A$, there exists $j \in \mathcal{H}(A)$ such that $i \wedge j \leq a \leq i \vee j$.

Theorem (F.-Jipsen 2022+):

Every centered residuated lattice embeds into a poset product of simple residuated lattices, and is therefore isomorphic to an algebra of antichain labelings.

Multipotence

Recall that a residuated lattice is multipotent if for all a there exists $n \in \mathbb{N}$ such that $a^{n+1} = a^n$.

Lemma (F.-Jipsen 2022+):

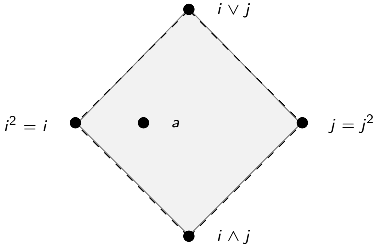
The follow are equivalent for a residuated lattice $\mathbf{A}$.

1. Every filter of $\mathbf{A}$ is central.
2. $\mathbf{A}$ is multipotent.

Proof: (1) $\Rightarrow$ (2). Let $a \in A$, and let $x = \mathrm{Fg}^{\mathbf{A}}(a)$. By assumption x is generated by the idempotents it contains, so there exists some $i \in \mathcal{H}(A)$ with $i \leq a$ and $i \in x$. On the other hand, since x is generated by $\{a\}$ there exists $n \in \mathbb{N}$ such that $a^n \leq i$. This implies that $i = a^n$, so $i = a^n$. It follows that $a^{n+1} = a^n$.

The square condition

The square condition: For every $i \in \mathcal{H}(A)$ and $a \in A$, there exists $j \in \mathcal{H}(A)$ such that $i \wedge j \leq a \leq i \vee j$.



The Blok-Ferreirim theorems

The Blok-Ferreirim theorem has had an impact in the theory of hoops and GBL-algebras, and roughly states that subdirectly irreducibles can be decomposed as a nested/ordinal sum with a totally ordered algebra on top.

When all the filters are central, as in centered residuated lattices, we can give a particularly nice form of this theorem due to the square condition:

Blok-Ferreirim Theorem for Centered Residuated Lattices (F.-Jipsen 2022+):

Let $\mathbf{A}$ be a subdirectly irreducible centered residuated lattice. Then there is a maximum element m of $\mathcal{H}(A) \setminus \{1\}$, and for all $a \in A$ we have $m \leq a$ or $a \leq m$.

Proof of the Blok-Ferreirim Theorem

Proof: Since $\mathbf{A}$ is subdirectly irreducible, so is $\mathcal{H}(\mathbf{A})$ (because all filters are central). Thus there exists a unique subcover m of 1 in $\mathcal{H}(\mathbf{A})$. Clearly, m is the maximum element of $\mathcal{H}(A) \setminus \{1\}$, so let $a \in A$. By the square condition, there exists $j \in \mathcal{H}(A)$ such that $m \wedge j \leq a \leq m \vee j$. By the choice of m, we have that $j = 1$ or $j \leq m$. If $j = 1$, then we get $m = m \wedge j \leq a$. If $j \leq m$, then $a \leq m \vee j = m$.

Sketch of the main proof

Let $\mathbf{A}$ be a centered residuated lattice. We will embed $\mathbf{A}$ in a poset product of simple residuated lattices.

Step 1: Let $(X, \subseteq)$ be the collection of values of $\mathbf{A}$ ordered by inclusion. Because all the filters of $\mathbf{A}$ are central, the lattice of filers of $\mathbf{A}$ is isomorphic to the lattice of filters of $\mathcal{H}(\mathbf{A})$ and we can just as well take the poset of values of $\mathcal{H}(\mathbf{A})$.

Step 2: For each $x \in X$, $\mathbf{A}/x$ is subdirectly irreducible because x is completely meet irreducible. The follow is not hard to show.

Lemma:

The class of centered residuated lattices is closed under quotients.

Hence, for each $x \in X$, $\mathbf{A}/x$ is a subdirectly irreducible centered residuated lattice.

Step 3: By the Blok-Ferreirim Theorem for centered residuated lattices, for each $\mathbf{A}/x$ there exists $m_x \in \mathcal{H}(\mathbf{A}/x)$ such that for all $a \in A/x$, $a \leq m_x$ or $m_x \leq a$. For each $x \in X$, define $A_x = \uparrow m_x$. Then A_x the universe of 0-free subalgebra of $\mathbf{A}/x$, so forms a residuated lattice $\mathbf{A}_x$.

Step 4: We claim that $\mathbf{A}$ embeds in $\prod_{(X,\subseteq)} \mathbf{A}_x$. The embedding is $a \mapsto [a](-)$, where for each $x \in X$,

$$[a](x) = \begin{cases} a/x & \text{if } m_x \leq a/x \\ 0 & \text{if } a/x < m_x. \end{cases}$$

The proof that $a \mapsto [a](-)$ is an embedding depends on the fact that $\mathcal{H}(\mathbf{A})$ is a central subalgebra of $\mathbf{A}$, together with $\mathbf{A}$ being multipotent (equivalent to each filter being central).

Varieties and logics

Centered residuated lattices don't form an especially nice class, and what we're interested in for logical purposes are varieties.

Definition:

For each $n \in \mathbb{N}$, let S_n denote the subvariety of residuated lattices axiomatized by:

- $a^n b = a^n \wedge b$.
- $a^n \to b^n = (a^n \to b^n)^2$.
- $a \leq b^n \vee (b^n \to a^n)$.

Further, for each $n \in \mathbb{N}$ denote by C_n the subvariety of S_n axiomatized by

$$(a \to b) \to (b \to a) = b \to a.$$

Concrete embedding theorems

Theorem (Jipsen-Montagna 2010):

For each $n \in \mathbb{N}$, the variety generated by poset products of simple n-potent MV-algebras chains is the variety of n-potent GBL-algebras.

Theorem (F.-Jipsen 2022+):

Let $n \in \mathbb{N}$.

- S_n is the variety generated by poset products of simple n-potent residuated lattices.
- C_n is the variety generated by poset products of simple n-potent MTL-algebras.

Part III:
Applications

A sketch of some applications

- As we discussed, representations by antichain labelings can be interpreted as Kripke-type semantics for substructural logics.
- In particular, the theorems of the last few slides indicate how to give Kripke semantics in terms of non-classical frames for the logics corresponding to each of the varieties S_n, C_n.
- Details about doing this in general can be found in F., Poset Products as Relational Models, Studia Logica 110:95–120 (2022), https://doi.org/10.1007/s11225-021-09956-z
- There's also a connection to modal logic that we've seen through the operator $\Box$.
- We'll outline the latter in the context of GBL-algebras, drawn from F. and Zuluaga, Some Modal and Temporal Translations of Generalized Basic Logic RAMiCS 2021, 176-191.
- Confined to GBL for clarity/interest, but easily applied to S_n, C_n etc.

The classical GMT translation

- The Gödel-McKinsey-Tarski translation connects intuitionistic logic (modeled by Heyting algebras) to the classical modal logic S4 (modeled by interior algebras).
- Recursively define a translation T from the language of intuitionsitic logic to modal logic by $T(p) = \Box p$ for any propositional variable p, $T(0) = 0$, $T(\varphi \star \psi) = T(\varphi) \star T(\psi)$ for $\star \in \{\wedge, \vee\}$, and $T(\varphi \to \psi) = \Box(\varphi \to \psi)$.
- Extend to sets of formulas in the obvious way: $T(\Gamma) = \{T(\varphi) : \varphi \in \Gamma\}$.

Theorem (Gödel, McKinsey, Tarski):

$\Gamma \vdash_{\mathbf{Int}} \varphi$ if and only if $T(\Gamma) \vdash_{\mathbf{S4}} T(\varphi)$.

S4MV-algebras

We can use the machinery of antichain labelings to give a fuzzy version of the GMT translation. The main algebraic models of our modal Łukasiewicz logic are as follows.

Definition:

We say that an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 0, 1, \{\Box\})$ is an S4MV-algebra provided that:

- $(A, \wedge, \vee, \cdot, \to, 0, 1)$ is an MV-algebra (BL-algebra with $(x \to 0) \to 0 = x$).
- $\Box$ is an interior operator and a $\{\wedge, \cdot, 0, 1\}$-endomorphism of $(A, \wedge, \vee, \cdot, \to, 0, 1)$.

S4MV-algebras are direct generalizations of the interior algebras that interpret classical S4; main difference is that $\Box$ is also assumed to preserve $\cdot$ (which is just $\wedge$ in the classical case).

Translating GBL

- In the embedding theorem for GBL-algebras, the poset product is a conuclear image of a direct product $\mathbf{B} = \prod_{x \in X} \mathbf{A}_x$ of a family of finite simple MV-algebras.
- Turns out that the conucleus $\Box$ satisfies the conditions so that $(\mathbf{B}, \Box)$ is an S4MV-algebra.
- Defining T as in the classical case, but stipulating that $T(\varphi \cdot \psi) = T(\varphi) \cdot T(\psi)$, we can prove:

Theorem (F.-Zuluaga 2021):

$\Gamma \vdash_{\mathbf{GBL}} \varphi$ if and only if $T(\Gamma) \vdash_{\mathbf{S4MV}} T(\varphi)$

- Actually, this is extended to temporal modalities in the paper.

On-going and future work

- Add topological content to what we've seen, extending Esakia duality to the substructural setting.
- Go beyond simple factors for more expressive representation theories.
- Further develop the connection to modal logic, going for a substructural Blok-Esakia theory of modal companions.

Thank you!