Interpolation in exchange-free logics

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Interpolation generally

- This talk is about interpolation, which deals with certain kinds of explanations for why given inferences hold.
- Craig interpolation property (CIP): If $\vdash \varphi \to \psi$, then there exists a formula δ such that $\mathrm{var}(\delta) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ and $\vdash \varphi \to \delta$ and $\vdash \delta \to \psi$.
- Various versions designed for particular applications: Uniform interpolation (databases), feasible interpolation (complexity theory), McMillan-style Craig interpolation (hardware and software verification), and so on.
- Deductive interpolation property (DIP): If $\Gamma \vdash \varphi$, then there exists a set of formulas Γ' such that $\mathrm{var}(\Gamma') \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi)$ and $\Gamma \vdash \Gamma'$ and $\Gamma' \vdash \varphi$.

The conventional wisdom

Broadly, interpolation is understood as a rather uncommon property.

- Exactly 7 consistent superintuitionistic logics with CIP/DIP, just 3 positive logics (Maksimova 1977).
- ≤ 38 normal extensions of S4 with CIP.
- Uncountably many extensions of Hájek's basic fuzzy logic without DIP (Montagna 2006).
- Positive results tend to use specialized methods and be fairly limited in scope.

The substructural environment

- Intuitionistic logic is a substructural logic.
- Generally these arise from dropping/relaxing some of the structural rules appearing in Gentzen's sequent calculus presentation of intuitionistic logic (exchange, weakening, contraction).
- Substructural logics encompass many logics arising independently:
  - Hájek's basic fuzzy logic and Łukasiewicz logic
  - The most prominent relevant logics
  - Linear logic and bunched implication logics
- Substructural logics can be formulated under the umbrella of extensions of the full Lambek calculus.

The full Lambek Calculus

Identity Axioms

$$\frac{}{\alpha \Rightarrow \alpha}\ (\textsc{id})$$

Cut Rule

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\textsc{cut})$$

Left Operation Rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \mathrm{e}, \Gamma_2 \Rightarrow \Delta}\ (\mathrm{e}\Rightarrow)$$

$$\frac{}{\mathrm{f} \Rightarrow}\ (\mathrm{f}\Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \beta/\alpha, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (/\Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \alpha\backslash\beta, \Gamma_3 \Rightarrow \Delta}\ (\backslash\Rightarrow)$$

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \cdot \beta, \Gamma_2 \Rightarrow \Delta}\ (\cdot\Rightarrow)$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_1$$

$$\frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \Delta}\ (\vee\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Rightarrow \mathrm{e}}\ (\Rightarrow\mathrm{e})$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{f}}\ (\Rightarrow\mathrm{f})$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow/)$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow\backslash)$$

$$\frac{\Gamma_1 \Rightarrow \alpha \quad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta}\ (\Rightarrow\cdot)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_1$$

$$\frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

Basic structural rules

$$\frac{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_2, \Pi_1, \Gamma_2 \Rightarrow \Delta}\ (\textsc{el})$$

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (\textsc{wl}) \qquad \frac{\Pi \Rightarrow}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (\textsc{wr})$$

$$\frac{\Gamma_1, \Pi, \Pi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (\textsc{cl})$$

$$\frac{\Gamma_1, \Pi_1, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}\ (\textsc{mingle})$$

Interpolation and exchange

Lots of success with DIP in the context of exchange without much systematic information:

- Lots of work from proof theory (Maehara, Ono, others).
- $\aleph_0$ extensions of Łukasiewicz logic with DIP (Di Nola-Lettieri 2000).
- Continuum-many extensions of FL + exchange with DIP, also for full linear logic (F.-Santschi 2023). Depends heavily on group theory.
- Previously thought that there may be no extension of FL lacking exchange with DIP (Gil Férez-Ledda-Tsinakis 2015).
- Example given in 2020 by Gil-Férez, Jipsen, Metcalfe.
- Several natural examples involving the law of excluded middle (F.-Galatos 2022).

Today's contribution

We will see that:

- There are continuum-many axiomatic extensions of FL without exchange that have DIP.
- All have the contraction and mingle rules, and are characteristic with respect to linearly ordered models (semilinear).
- Among axiomatic extensions of falsum-free FL + contraction + mingle + exchange + semilinearity, only 60 with DIP.

Part I:
The set-up

Logics without exchange

- Note that exchange is derivable in the presence of contraction + left weakening.
- So, if we want to study extensions of FL without exchange while doing minimal mutilation to the intuitionistic framework, we can't keep both contraction and weakening.
- Natural solution: Replace one of contraction or weakening by a slightly less powerful rule.
- Here we replace weakening by the mingle rule.
- We thus focus on FL_{cm}, full Lambek calculus + contraction + mingle.
- We also consider the variant without falsum f.

Algebraic semantics

- Key methodology: Algebraization of the consequence relation of FL.
- Algebraization gives mutually inverse, back-and-forth translations between a consequence relation and the equational consequence relation of some class of algebraic models (in our case, residuated lattices).
- Transfer many properties by bridge theorems:
    - Local deduction theorems correspond to the congruence extension property.
    - With the above, DIP corresponds to the amalgamation property.

Residuated lattices

A residuated lattice is an algebraic structure of the form $(A, \wedge, \vee, \cdot, \backslash, /, e)$ where

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, e)$ is a monoid, and
- for all $x, y, z \in A$,

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y.$$

We use all the expected terminology: Commutative, idempotent, totally ordered, linear, etc.

Semilinear: Subalgebra of a direct product of totally ordered residuated lattices.

Note that despite the adjunction condition, residuated lattices form a variety (equational class). Subvarieties of residuated lattices correspond exactly with axiomatic extensions of FL without falsum.

Some corresponding properties

Because of algebraization, there's a back-and-forth dictionary of concepts:

- Exchange corresponds to commutativity $xy = yx$.
- (Left) weakening correspond to integrality $x \leq e$.
- Contraction corresponds to the square-increasing law $x \leq x^2$.
- Mingle corresponds to the square-decreasing law $x^2 \leq x$.
- So, contraction + mingle corresponds to multiplication being idempotent $x^2 = x$.
- To study axiomatic extensions of positive FL + contraction + mingle, we can study varieties (equational classes) of idempotent residuated lattices.
- Semilinearity corresponds to the communication rule.

Amalgamation

Definition:

Let $\mathcal{K}$ be a class of algebraic structures. A span in $\mathcal{K}$ is a quintuple (A, B, C, f, g), where $A, B, C \in \mathcal{K}$ and $f \colon A \to B$, $g \colon A \to C$ are embeddings. We say that $\mathcal{K}$ has the amalgamation property (or AP) if for every span (A, B, C, f, g) in $\mathcal{K}$ there exists $D \in \mathcal{K}$ and embeddings $f' \colon B \to D$ and $g' \colon C \to D$ such that $f' \circ f = g' \circ g$.

$$\begin{tikzcd} & \mathbf{C} \\ \mathbf{A} \arrow[ur, hook, "f_C"] \arrow[dr, hook, "f_B"'] & \\ & \mathbf{B} \end{tikzcd}$$

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$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "g_C"] & \\
\mathbf{A} \arrow[ur, hook, "f_C"] \arrow[dr, hook, "f_B"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, hook, dashed, "g_B"'] &
\end{tikzcd}
$$

Part II:
The case without exchange

The work plan

- To get continuum-many axiomatic extensions of FL $+$ contraction $+$ mingle with the DIP, it's enough to come up with continuum-many varieties of semilinear idempotent residuated lattices with the amalgamation property.
- We're inspired by Galatos 2005, which gives continuum-many atoms in the lattice of subvarieties of semilinear idempotent residuated lattices (logics with no non-trivial extensions).
- We'll show that each of Galatos's varieties have the amalgamation property.
- This involves four ingredients: The nested sum construction of residuated lattices, the symbolic dynamics of bi-infinite words, tools from first-order model theory, and new characterizations of the AP.

Starting out

Suppose $S \subseteq \mathbb{Z}$. We define an algebra on

$$A_S = \{a_i : i \in \mathbb{Z}\} \cup \{b_j : j \in \mathbb{Z}\} \cup \{e\}.$$

Order the elements of A_S by setting $b_i < b_j < e < a_k < a_l$ if and only if $i, j, k, l \in \mathbb{Z}$ with $i < j$ and $l < k$. Further, for $i, j \in \mathbb{Z}$ define $a_i a_j = a_{\min\{i,j\}}$, $b_i b_j = b_{\min\{i,j\}}$, and

$$a_i b_j = \begin{cases} a_i & \text{if } i < j \text{ or } i = j \in S \\ b_j & \text{if } i > j \text{ or } i = j \notin S \end{cases}$$

,

$$b_j a_i = \begin{cases} b_j & \text{if } j < i \text{ or } i = j \in S \\ a_i & \text{if } j > i \text{ or } i = j \notin S \end{cases}$$

We stipulate that e is a multiplicative identity and define residuals $\backslash$ and $/$ in the usual way. The residuated lattice obtained in this way is denoted by $\mathbf{A}_S$ and the variety it generates is V_S.

Bi-infinite words

Definition:

A word over $\{0,1\}$ is a function $w\colon A \to \{0,1\}$, where A is some subinterval of $\mathbb{Z}$. A word is finite if $|A|$ is finite and bi-infinite if $A = \mathbb{Z}$. We say that a finite word $v\colon A \to \{0,1\}$ is a subword of a word w if there exists an integer k such that $v(i) = w(i+k)$ for all $i \in A$. The characteristic function w_S of a subset $S \subseteq \mathbb{Z}$ is an example of a bi-infinite word.

Definition:

We define a pre-order $\sqsubseteq$ on the set of all bi-infinite words by setting $w_1 \sqsubseteq w_2$ if and only if every finite subword of w_1 is a subword of w_2. For bi-infinite words w_1, w_2, we write $w_1 \cong w_2$ if and only if $w_1 \sqsubseteq w_2$ and $w_2 \sqsubseteq w_1$.

Fact:

There are continuum-many pairwise incomparable minimal bi-infinite words.

Constructing the subvarieties

- For each $S \subseteq \mathbb{Z}$, we can consider S as a bi-infinite word by identifying it with its characteristic function w_S.
- If w_S is minimal, then V_S gives an atom in the lattice of subvarieties of semilinear idempotent residuated lattices.
- The cardinality result for atoms follows from the fact that there continuum-many pairwise incomparable minimal bi-infinite words.

Nested sums

- The nested sum extends the well-known ordinal sum construction used for Hájek's basic logic.
- It is technical to state correctly, but it amounts to replacing the identity element e in a residuated lattice $\mathbf{A}$ by another residuated lattice $\mathbf{B}$.
- This can only be done for some residuated lattices, but it turns out that the algebra $\mathbf{A}_S$ are admissible.

The key lemma

Lemma:

Suppose that $S \subseteq \mathbb{Z}$.

1. $\mathbb{HSP}_U(\mathbf{A}_S)$ is the class of totally ordered members of V_S. In particular, $\mathbb{HSP}_U(\mathbf{A}_S)$ consists of the finitely subdirectly irreducible members of V_S.
2. If w_S is minimal, then $\mathbb{HSP}_U(\mathbf{A}_S)$ is closed under nested sums. In particular, the finitely subdirectly irreducible members of V_S are exactly nested sums of members of $\mathsf{K}_S = \mathbb{I}(\{\mathbf{A}_T : w_T \sqsubseteq w_S\})$.

The proof is a technical argument using ultraproducts, and invokes the fact that every algebra embeds into an ultraproduct of its finitely generated subalgebras.

Amalgamation of chains

Lemma:

Suppose that $S \subseteq \mathbb{Z}$ is such that w_S is minimal. Then the class of totally ordered members in V_S has the amalgamation property.

The proof involves decomposing each chain in a given span into a nested sum of its 1-generated subalgebras (by F.-Galatos 2022), and then collecting 1-generated subalgebras. Because the totally ordered members are closed under nested sums by the previous lemma, these can be collected into an amalgam by taking the nested sum.

From chains upward

This doesn't quite prove that the varieties V_S, w_S minimal, have the AP. For this, we need to extend the AP from chains:

Theorem (F.-Metcalfe 2022)

Suppose V is a congruence-distributive variety with the congruence extension property, and that the class of finitely subdirectly irreducibles in V is closed under taking subalgebras. Then if the class of finitely subdirectly irreducibles in V has the amalgamation property, so does V.

Theorem (F.-Galatos 2022)

The variety of semilinear idempotent residuated lattices has the congruence extension property.

Centrality

- Let $x^* = x\backslash e \vee e/x$. It follows from (F.-Galatos 2022) that if $\mathbf{A}$ is a idempotent residuated chain and $x \in A$, then x fails to commute with at most one element and that element is x^*.
- Thus $xx^* = x^*x \Rightarrow x = e$ expresses that the only central element in an idempotent residuated chain $\mathbf{A}$ is e.
- We can show that if w_S is minimal, then each member of V_S satisfies this quasiequation.
- We will call these exchange-free, and use the same terminology for the corresponding logics.

Main theorem

We have proven:

Theorem:

There are continuum-many axiomatic extensions of FL $+$ contraction $+$ mingle $+$ semilinearity with the DIP. Each of these axiomatic extensions is exchange-free and has no non-trivial extensions.

Part III:
Further results

Extensions without the DIP

Leveraging some known failures of amalgamation in varieties of semilinear idempotent residuated lattices, we can also obtain the following:

Theorem:

There are continuum-many axiomatic extensions of FL + contraction + mingle + semilinearity refuting the exchange rule, but without the DIP.

Returning to exchange

- If we add exchange back into the picture, the constructions available in the non-commutative case can't be simulated.
- Structural results on commutative idempotent residuated chains, plus application of one-sided amalgamation, gives:

Theorem:

There are exactly 60 axiomatic extensions of falsum-free FL + exchange + contraction + mingle + semilinearity with the DIP.

- The proof of this amounts to a technical counting argument, not so different from Maksimova's result on intuitionistic logic (hinges on forbidden configurations).

Adding falsum

- The picture doesn't change that much if we return the falsity constant f to the signature.
- There are still finitely many extensions with DIP in the case with exchange + contraction + mingle + semilinearity.
- But there are many more, and counting them is rather tedious.
- Main idea is that the placement of f in linearly ordered models determines how to decompose these models as a nested sum.

Conclusion

- Combined with the results of (F.-Santschi 2023), this work resolves most of the questions about the number of extensions with DIP for FL $+$ basic structural rules.
- Most interesting open questions involve the weakening rule.
- The tools to get these results are frustratingly diverse, and also require a lot of technology that didn't exist just a few years ago.
- Could pose similar questions about Craig interpolation, uniform interpolation, and so forth.
- This would require new basic tools from, e.g., universal algebra and order-algebraizable logics.

Thank you!