Interpolation in Substructural Logics II: Logics without Exchange

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Introduction

- Santschi's talk: Substructural logics with the exchange rule,

$$\frac{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_2, \Pi_1, \Gamma_2 \Rightarrow \Delta.}$$

- This talk: The role of the exchange rule in limiting the scope of (deductive) interpolation.
- There has been quite a bit of work done on exchange and interpolation:
  - Many logics with exchange that have deductive interpolation.
  - Once an open question whether every substructural logic with deductive interpolation derives the exchange rule (Gil-Férez–Ledda–Tsinakis 2015).
  - Later shown to not be the case by a counterexample (Gil-Férez–Jipsen–Metcalfe 2020).

Basic structural rules

$$\frac{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_2, \Pi_1, \Gamma_2 \Rightarrow \Delta}\ (e) \qquad \frac{\Gamma_1, \Pi, \Pi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (c)$$

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (i) \qquad \frac{\Pi \Rightarrow}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (o)$$

$$\frac{\Gamma_1, \Pi_1, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}\ (m)$$

Logics without exchange

- Note that exchange is derivable in the presence of contraction + left weakening (i).
- So, if we want to study extensions of FL without exchange while doing minimal mutilation to the intuitionistic framework, we need can't keep both contraction and weakening.
- Natural solution: Replace one of contraction or weakening by a slightly less powerful rule.
- Here we replace weakening by the mingle rule.
- $\mathsf{SemRL_{cm}} = \mathsf{FL} + (e \approx f) + (c) + (m) + \textit{semilinearity}$.

Outline

Theorem A

1. There are continuum-many varieties of idempotent semilinear residuated lattices that have the amalgamation property and contain non-commutative members.
2. There are continuum-many axiomatic extensions of $\mathsf{SemRL}_{\mathsf{cm}}$ that have the deductive interpolation property in which the exchange rule is not derivable.

Theorem B

1. There are exactly sixty varieties of commutative idempotent semilinear residuated lattices that have the amalgamation property.
2. There are exactly sixty axiomatic extensions of $\mathsf{SemRL}_{\mathsf{ecm}}$ that have the deductive interpolation property.

Residuated lattices

A residuated lattice is an algebraic structure of the form $(A, \wedge, \vee, \cdot, \backslash, /, e)$ where

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, e)$ is a monoid, and
- for all $x, y, z \in A$,

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y.$$

We use all the expected terminology: Commutative, idempotent, totally ordered, linear, etc.

Semilinear: Subalgebra of a direct product of totally ordered residuated lattices.

Bi-infinite words

Definition:

A word over $\{0,1\}$ is a function $w\colon A \to \{0,1\}$, where A is some subinterval of $\mathbb{Z}$. A word is finite if $|A|$ is finite and bi-infinite if $A = \mathbb{Z}$. We say that a finite word $v\colon A \to \{0,1\}$ is a subword of a word w if there exists an integer k such that $v(i) = w(i+k)$ for all $i \in A$. The characteristic function w_S of a subset $S \subseteq \mathbb{Z}$ is an example of a bi-infinite word.

Definition:

We define a pre-order $\leq$ on the set of all bi-infinite words by setting $w_1 \leq w_2$ if and only if every finite subword of w_1 is a subword of w_2. For bi-infinite words w_1, w_2, we write $w_1 \cong w_2$ if and only if $w_1 \leq w_2$ and $w_2 \leq w_1$.

Fact:

There are uncountably many pairwise incomparable minimal bi-infinite words.

Suppose $S \subseteq \mathbb{Z}$. We define an algebra on

$$A_S = \{a_i : i \in \mathbb{Z}\} \cup \{b_j : j \in \mathbb{Z}\} \cup \{e\}.$$

Order the elements of A_S by setting $b_i < b_j < e < a_k < a_l$ if and only if $i, j, k, l \in \mathbb{Z}$ with $i < j$ and $l < k$. Further, for $i, j \in \mathbb{Z}$ define $a_i a_j = a_{\max\{i,j\}}$, $b_i b_j = b_{\min\{i,j\}}$, and

$$a_i b_j = \begin{cases} a_i & \text{if } i < j \text{ or } i = j \in S \\ b_j & \text{if } i > j \text{ or } i = j \notin S \end{cases}$$

,

$$b_j a_i = \begin{cases} b_j & \text{if } j < i \text{ or } i = j \in S \\ a_i & \text{if } j > i \text{ or } i = j \notin S \end{cases}$$

We stipulate that e is a multiplicative identity and define residuals $\backslash$ and $/$ in the usual way. The residuated lattice obtained in this way is denoted by $\mathbf{A}_S$.

Theorem A

Theorem A

1. There are continuum-many varieties of idempotent semilinear residuated lattices that have the amalgamation property and contain non-commutative members.
2. There are continuum-many axiomatic extensions of $\mathrm{SemRL_{cm}}$ that have the deductive interpolation property in which the exchange rule is not derivable.

Proof Sketch: Let w_S be a minimal bi-infinite word. We want to show that $\mathbb{V}(\mathbf{A}_S)$ has the amalgamation property.

Using general results on amalgamation (Fussner-Metcalfe 2023), it is enough to show that spans of finitely subdirectly irreducible members (chains) in $\mathbb{V}(\mathbf{A}_S)$ have amalgams. This works because these varieties have the congruence extension property (Fussner-Galatos 2023).

More on Theorem A

Let

$$\mathsf{K}_S = \mathbb{I}(\{\mathbf{A}_T \mid w_T \leq w_S\}).$$

The hard part of this is to show that the members of $\mathbb{HSP}_u(\mathbf{A}_S)$ are exactly nested sums of members of K_S.

Then if $(f_B \colon \mathbf{A} \to \mathbf{B}, f_C \colon \mathbf{A} \to \mathbf{C})$ is a span of finitely subdirectly irreducible members of $\mathbb{V}(\mathbf{A}_S)$, the amalgam can be found by looking at nested sum decompositions of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, collecting common summands, and taking a nested sum of all the summands.

That there are continuum-many varieties $\mathbb{V}(\mathbf{A}_S)$ for w_S minimal comes from the fact that there are continuum-many pairwise incomparable minimal bi-infinite words.

Reinstating exchange

Theorem B

1. There are exactly sixty varieties of commutative idempotent semilinear residuated lattices that have the amalgamation property.
2. There are exactly sixty axiomatic extensions of $\mathsf{SemRL}_{\mathsf{ecm}}$ that have the deductive interpolation property.

Encoding bi-infinite words in the algebras $\mathbf{A}_S$ depends on their non-commutativity. If we look at commutative semilinear idempotent residuated lattices, the situation is much different.

A key difference is that commutative semilinear idempotent residuated lattices are locally finite.

This allows us to consider only spans of finite residuated lattice chains, and reduces the problem to a combinatorial one.

Model completions

It has long been known that the first-order theory of a locally finite variety has a model completion if and only if the variety has the amalgamation property. So, as an immediate consequence of Theorem B, we get:

Theorem C

There are exactly sixty varieties of commutative idempotent semilinear residuated lattices whose first-order theories have a model completion.

Some final remarks

- If we drop the requirement that $e = f$, then the combinatorial arguments involved in counting varieties of commutative semilinear idempotent residuated lattices with amalgamation break down.
- In this case, we can still show that there are finitely many varieties with amalgamation, but there are many more (12,000,000 is a lower bound).
- There are still some prominent open cases. Most notably, we don't know whether there are finitely many varieties of idempotent commutative residuated lattices that have amalgamation.

Thank you!

References

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