Categories of Residuated Lattices

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Abstract

We present dual variants of two algebraic constructions of certain classes of residuated lattices: The Galatos-Raftery construction of Sugihara monoids and their bounded expansions, and the Aguzzoli-Flaminio-Ugolini quadruples construction of srDL-algebras. Our dual presentation of these constructions is facilitated by both new algebraic results, and new duality-theoretic tools. On the algebraic front, we provide a complete description of implications among nontrivial distribution properties in the context of lattice-ordered structures equipped with a residuated binary operation. We also offer some new results about forbidden configurations in lattices endowed with an order-reversing involution. On the duality-theoretic front, we present new results on extended Priestley duality in which the ternary relation dualizing a residuated multiplication may be viewed as the graph of a partial function. We also present a new Esakia-like duality for Sugihara monoids in the spirit of Dunn's binary Kripke-style semantics for the relevance logic R-mingle.

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For me, mathematics is a great cooperative project going back to Euclid. Collaboration is its lifeblood, and therefore I thank first the fantastic researchers who I have the honor of counting as collaborators on topics related to this thesis. Chief among them is my doctoral supervisor Nick Galatos, who introduced me to many of these topics, and, more importantly, taught me how to *really* think about them. My other collaborators also have my deepest gratitude: Peter Jipsen, whose deep insight and vision for mathematics have been a profound inspiration to me; Alessandra Palmigiano, who welcomed me into her research community and kindly hosted me at her home when I was writing this thesis; and Sara Ugolini, whose good sense and technical know-how have been such an edifying force in my life.

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Chapter 1

Introduction

In the mathematical study of reasoning, algebraic logic is among the dominant paradigms in part because it provides a powerful framework for comparing diverse propositional logical systems. When algebraization of a propositional logic is achievable, it reifies the logic by interpreting it in terms of tangible algebraic structures, providing semantic content. This process often represents a vast simplification of the logic, effectively permitting us to ignore much of the syntactic complexity of formal reasoning and thereby see important features that would have otherwise remained obscure. Surprisingly, many interesting fine-grained distinctions survive this simplification, at least in well-behaved cases. For instance, an algebraizable propositional logic is so closely bound to its equivalent algebraic semantics (see [5]) that its lattice of axiomatic extensions is dually-isomorphic to the lattice of subvarieties of its algebraic semantics.

Perhaps even more impressively, the *metalogical* properties of a propositional logic may often be faithfully recast in terms of its algebraic semantics. For one well-known example: Under appropriate technical hypotheses, a logic possesses the interpolation property if and only if its equivalent algebraic semantics has the amalgamation property (see [16]). Significantly, the amalgamation property is a *categorical* property: It depends only on the ambient category of algebraic structures and not on any internal features of the algebras (not even those as seemingly intrinsic to the setting as their type). This underscores the importance of categorical properties of logics' algebraic semantics, and in particular relationships among such categories.

This thesis is about such relationships, especially those that present themselves as algebraic constructions connecting one kind of algebraic semantics to another. Often these constructions give categorical equivalences between varieties of logic algebras, and often these constructions are spectacularly complicated. The *leitmotif* of the present work is the repackaging of this complexity so as to reveal hidden aspects of constructions on algebras of logic. Sometimes we achieve this aim by purely algebraic means (e.g., in Chapter 5). More often, we rely on *topological dualities* for lattice-based algebras to recast constructions in more pictorial and transparent terms. Among other benefits, topological dualities afford representations of algebraic structures that inform our analysis of the algebras themselves. Sometimes the insight contributed by dual representations of algebras allows us to simplify their theory in a manner that implicates new duality-theoretic results, initiating a mutually-supporting feedback loop between algebraic analysis and duality theory.

We sketch our work as follows. Chapter 2 introduces the algebraic environment in which we will work in the following chapters, in particular setting out needed background on residuated lattices and lattices with involution (aka i-lattices). The former provide the equivalent algebraic semantics for substructural logics, whereas the latter provide a general and flexible framework for thinking about negation in nonclassical logic.[1] Residuated lattices and i-lattices are married in our discussion

[1] Although we will neglect the syntactic aspects of the logics corresponding to these classes of algebras, it is nevertheless conceptually valuable to think of them in logical rather than purely algebraic terms.

of involutive residuated lattices, and we also discuss two classes of negation-bearing residuated structures (i.e., Sugihara monoids and srDL-algebras) that will provide case studies for later work. In addition to providing an exposition of the theory of these algebras, Chapter 2 offers some new results about varieties of residuated structures satisfying certain distributive laws (see Section 2.1.1), as well as some new results regarding forbidden configurations in lattices with involution (see Section 2.2.1).

Chapter 3 offers background material on duality theory. This includes an introduction to natural dualities, as well as generalities regarding the more classical Stone-Priestley and Esakia dualities for distributive lattices and Heyting algebras. We also discuss extended Stone-Priestley duality, an augmentation of Stone-Priestley duality that accounts for the addition of residuated operations and involution. Almost all of this chapter consists of well-known preliminary material, but Sections 3.3 and 3.4.1 provide new results regarding the omission of lattice bounds from the algebraic signature. This provides a natural duality for distributive i-lattices satisfying the *normality* condition $x \wedge \neg x \leqslant y \vee \neg y$, as well as a duality for bottom-free reducts of monoidal t-norm based logic algebras (aka GMTL-algebras).

Chapter 4 explores the phenomenon of *functionality* in the context of extended Stone-Priestley duality. Although residuated operations are typically presented on dual spaces by a ternary relation, under appropriate hypotheses this ternary relation may be interpreted as a partial binary operation. This is the case in the context of Sugihara monoids and srDL-algebras, for example, and we lay out some of the theory of functional extended Stone-Priestley duality for the pertinent classes of algebras. We also provide a more abstract treatment of the functionality phenomenon, working in the canonical extension of certain distributive lattices with operators in order to obtain a new perspective. The results of this chapter come from the author's [26, 27].

$$
\begin{tikzcd}
\mathsf{SM}_\perp \arrow[r, shift left, "\mathcal{S}"] \arrow[d, shift left, "(-)_{\bowtie}"] \arrow[dr, shift left, "\mathcal{D}"] & \mathsf{SM}_\perp^\tau \arrow[l, shift left, "\mathcal{A}"] \arrow[d, shift left, "(-)_{\bowtie}"] \\
\mathsf{bGA} \arrow[r, shift left, "\mathcal{S}"] \arrow[u, shift left, "(-)^{\bowtie}"] & \mathsf{SS} \arrow[l, shift left, "\mathcal{A}"] \arrow[u, shift left, "(-)^{\bowtie}"] \arrow[ul, shift left, "\mathcal{E}"]
\end{tikzcd}
$$

Figure 1.1: Various equivalences among the categories in the vicinity of bounded Sugihara monoids $\mathsf{SM}_\perp$.

Chapter 5 inaugurates our effort to use previous chapters' duality-theoretic machinery to simplify constructions. This chapter recalls the Galatos-Raftery construction [30, 31] of Sugihara monoids (i.e., idempotent distributive commutative residuated lattices with a compatible involution) from certain enriched relative Stone algebras (i.e., semilinear residuated lattices where multiplication coincides with the lattice meet). Together with the two following chapters, Chapter 5 offers a presentation of the Galatos-Raftery construction on certain structured topological spaces. This dual variant of the Galatos-Raftery construction implicates both the Davey-Werner duality for normal distributive i-lattices, as well as the extended Priestley duality specialized to Sugihara monoids (or, more precisely, their bounded expansions). The web of categories tied together by the Galatos-Raftery construction, its dual, and these topological dualities provides a multifaceted description of categories providing semantics for the relevance logic R-mingle, as equipped with Ackermann constants (see Figure 1.1). Chapter 5 contributes to this project through algebraic work that simplifies the Galatos-Raftery construction, anchoring it in representations tailored to accentuate the i-lattice structure of Sugihara monoids. This is necessary preprocessing for the duality-theoretic applications in subsequent chapters. These results originally appeared in the author's [24].

Chapter 6 introduces a new duality for Sugihara monoids that is focused on their i-lattice structure. This duality shares much in common with Esakia's celebrated duality for Heyting algebras [21], and is obtained by restricting the Davey-Werner duality for i-lattices to those i-lattices that appear as reducts of Sugihara monoids. The duality of this chapter provides the diagonal of Figure 1.1, and it originally appears in the author's [24].

Chapter 7 utilizes the results of Chapters 5 and 6 to provide our dual variant of the Galatos-Raftery construction. This dual variant is vastly more transparent and pictorial than its algebraic counterpart, and completes our study of Sugihara monoids. The results of this chapter come from [24].

Moving from Sugihara monoids to our second case study, Chapter 8 provides a dual variant of the Aguzzoli-Flaminio-Ugolini construction [1] of large classes of monoidal t-norm logic algebras from their Boolean skeletons and radicals. The dual construction shares much in common with our dual variant of the Galatos-Raftery construction, and makes plain conceptual similarities between the two constructions. Moreover, our dual variant of the Aguzzoli-Flaminio-Ugolini construction reveals hidden aspects of the order-theoretic structure of the algebraic version of the construction, while presenting the monoidal/residuated content of the construction in a much simpler fashion. This work is drawn from the author's [27].

Chapter 2

Residuated algebraic structures

This preliminary chapter lays out background regarding the algebraic structures pertinent to the work to follow. Much of the material presented here is folklore, and will be summarized without proof. For a more leisurely presentation of the theory of residuated structures, we refer the reader to the standard monograph [29] (but see also [8], which provides a different perspective).

We strive to make our treatment as self-contained as possible, but presume familiarity with the elements of lattice theory and universal algebra. For general information on these subjects, we refer to the texts [18], [7], and [9]. Our results are often framed in the language of category theory, information on which may be found in [3] and [43]. We defer providing background on duality theory until Chapter 3.

Although the primary purpose of this chapter is to recall preliminaries, some material is new. The results on distributive laws in Section 2.1.1 are the author's own [25], as are the results on forbidden configurations in i-lattices in Section 2.2.1. In these cases, we offer a more thorough discussion and furnish proofs where relevant and informative.

2.1 Residuated structures

A *residuated binar*[2] is an algebra $(A, \wedge, \vee, \cdot, \backslash, /)$ of type $(2, 2, 2, 2, 2)$, where $(A, \wedge, \vee)$ is a lattice, and for all $x, y, z \in A$,

$$y \leqslant x\backslash z \iff x \cdot y \leqslant z \iff x \leqslant z/y.$$

The latter demand is often called the *law of residuation*. When $\cdot$ is a binary operation on some lattice, $\cdot$ is said to be *residuated* when there exist binary operations $\backslash$ and $/$ for which the law of residuation holds. The division operations $\backslash$ and $/$ are called the *residuals* of the multiplication $\cdot$.

In order to promote readability, we often abbreviate $x \cdot y$ by xy. We will also adopt the convention that $\cdot$ binds more strongly than $\backslash, /$, which in turn bind more strongly than $\wedge, \vee$.

Proposition 2.1.1. *[29, Theorem 3.10] Let $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /)$ be a residuated binar.*

1. *Multiplication preserves existing joins in each argument, i.e., if $X, Y \subseteq A$ and $\bigvee X$ and $\bigvee Y$ exist, then*

$$\bigvee X \cdot \bigvee Y = \bigvee \{xy : x \in X, y \in Y\}.$$

2. *Divisions preserve all existing meets in the numerator, and convert all existing joins in the denominator to meets, i.e., if $X, Y \subseteq A$ and $\bigvee X$, $\bigwedge Y$ exist, then for any $z \in A$ each of $\bigwedge_{x \in X} x\backslash z$, $\bigwedge_{x \in X} z/x$, $\bigwedge_{y \in Y} z\backslash y$, and $\bigwedge_{y \in Y} y/z$ exists*

[2] A *binar* is a set equipped with a binary operation. Binars are also commonly called *groupoids* or *magmas*.

and

$$z\backslash(\bigwedge Y) = \bigwedge_{y\in Y} z\backslash y, \qquad (\bigwedge Y)/z = \bigwedge_{y\in Y} y/z.$$

$$(\bigvee X)\backslash z = \bigwedge_{x\in X} x\backslash z, \qquad z/(\bigvee X) = \bigwedge_{x\in X} z/x.$$

3. $x\backslash z = \max\{y \in A : xy \leqslant z\}$ *and* $z/y = \max\{x \in A : xy \leqslant z\}$

Remark 2.1.2. Proposition 2.1.1 has a partial converse. Specifically, if $(A, \wedge, \vee)$ is a complete lattice endowed with an additional binary operation $\cdot$, then $\cdot$ is residuated provided that it distributes over arbitrary joins in each coordinate. For finite lattices (and somewhat more generally), it suffices for $\cdot$ to distribute over binary joins.

The following is an easy consequence of Proposition 2.1.1.

Proposition 2.1.3. *[29, Corollary 3.14] Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /)$ *be a residuated binar. Then* $\cdot$ *is isotone in each coordinate, and* $\backslash$ *and* $/$ *are isotone in their numerators and antitone in their denominators. Moreover,* $\mathbf{A}$ *satisfies the following identities.*

$(\cdot\vee)$ $x(y \vee z) = xy \vee xz.$

$(\vee\cdot)$ $(x \vee y)z = xz \vee yz.$

$(\backslash\wedge)$ $x\backslash(y \wedge z) = x\backslash y \wedge x\backslash z.$

$(\wedge/)$ $(x \wedge y)/z = x/z \wedge y/z.$

$(/\vee)$ $x/(y \vee z) = x/y \wedge x/z.$

$(\vee\backslash)$ $(x \vee y)\backslash z = x\backslash z \wedge y\backslash z.$

Observe that the law of residuation is not *prima facie* an equational condition. However, one may show that residuated binars form a finitely-based variety.

Residuated binars need not have a multiplicative neutral element. If $\mathbf{A}$ is a residuated binar with a multiplicative neutral element e, we say that an expansion of $\mathbf{A}$ by a constant designating e is *unital*. If $\mathbf{A}$ is a residuated binar with multiplicative neutral element e, then we say that $\mathbf{A}$ is *integral* if it satisfies $x \leqslant e$. The following gives some properties of integral residuated binars.

Proposition 2.1.4. *[29, see, e.g., Lemma 3.15] Let $\mathbf{A}$ be an integral residuated binar with multiplicative neutral element e. Then $\mathbf{A}$ satisfies the following identities.*

1. $xy \leqslant x \wedge y$.
2. $y \leqslant x\backslash y$.
3. $x \leqslant x/y$.
4. $x\backslash x = x/x = e$.

A residuated binar may also lack universal bounds with respect to its underlying lattice order. However, if $\mathbf{A}$ is a residuated binar with least element $\bot$, then $\mathbf{A}$ satisfies the equations $x \cdot \bot = \bot \cdot x = \bot$. Consequently, $\mathbf{A}$ also has a greatest element $\top$ and $\top = \bot\backslash\bot = \bot/\bot$. We refer to an expansion of a residuated binar by a constant designating a least element $\bot$ as a *bounded* residuated binar. Note that bounded residuated binars are term-equivalent to the expansions of residuated binars by constants designating both least and greatest elements.

We say that an expansion of a residuated binar $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /)$ is *modular*, *distributive*, *complemented*, or *Boolean* provided that $(A, \wedge, \vee)$ is. Likewise, we say that $\mathbf{A}$ is *commutative*, *associative*, or *idempotent* provided that $(A, \cdot)$ is. Note that if $\mathbf{A}$ is a commutative residuated binar, then $\mathbf{A}$ satisfies $x\backslash y = y/x$. In this event, we denote the common value of $x\backslash y$ and y/x by $x \to y$. For commutative residuated binars, we work with the term-equivalent signature involving the single binary operation $\to$ rather than $\backslash$ and $/$.

We will call an associative residuated binar a *residuated semigroup*. Unital residuated semigroups are called *residuated lattices*, and comprise the most important and thoroughly-studied class of residuated structures. We will return to residuated lattices in Section 2.3.

If K is a class of similar algebras with lattice reducts, we say that $\mathbf{A} \in \mathsf{K}$ is K-*semilinear* if $\mathbf{A}$ is a subalgebra of a product of linearly-ordered algebras in K, and if K is clear from context we simply say that $\mathbf{A}$ is *semilinear*. Since chains are distributive lattices, the lattice reduct of a semilinear algebra is always distributive.

2.1.1 Distributive laws

Owing to Proposition 2.1.1 and Remark 2.1.2, one may think of the law of residuation as articulating a kind of strong distributive property. However, neither lattice distributivity nor any of the identities

$$x(y \wedge z) = xy \wedge xz \tag{$\cdot\wedge$}$$

$$(x \wedge y)z = xz \wedge yz \tag{$\wedge\cdot$}$$

$$x\backslash(y \vee z) = x\backslash y \vee x\backslash z \tag{$\backslash\vee$}$$

$$(x \vee y)/z = x/z \vee y/z \tag{$\vee/$}$$

$$(x \wedge y)\backslash z = x\backslash z \vee y\backslash z \tag{$\wedge\backslash$}$$

$$x/(y \wedge z) = x/y \vee x/z \tag{$/\wedge$}$$

hold in the variety of residuated binars (cf. the distributive laws in Proposition 2.1.3). Blount and Tsinakis showed in [6] that in a residuated lattice satisfying distributivity at e, viz.

$$(x \vee y) \wedge e = (x \wedge e) \vee (y \wedge e),$$

the equations $e \leqslant x/y \vee y/x$, $(/\wedge)$, and $(\vee/)$ are equivalent. Likewise, in the presence of distributivity at e, the equations $e \leqslant y\backslash x \vee x\backslash y$, $(\wedge\backslash)$, and $(\backslash\vee)$ are equivalent. Semilinear residuated lattices satisfy all of these nontrivial distributive laws, but a residuated lattice may satisfy all six of these identities but fail to be semilinear (this is true of lattice-ordered groups, for example).

The goal of this section is to understand inferential relationships among these six nontrivial distributive laws, a typical instance of which is given in the following.

Proposition 2.1.5. *Let* $\mathbf{A}$ *be a distributive residuated binar. Then if* $\mathbf{A}$ *satisfies both* $(\vee/)$ *and* $(\wedge\backslash)$, $\mathbf{A}$ *also satisfies* $(\backslash\vee)$.

Proof. Note that $(\wedge\backslash)$ is equivalent to the identity

$$(x \wedge y)\backslash(z \wedge w) \leqslant x\backslash z \vee y\backslash w,$$

whereas $(\backslash\vee)$ is equivalent to the identity

$$(x \vee y)\backslash(z \vee w) \leqslant x\backslash z \vee y\backslash w.$$

Let $u \leqslant (x \vee y)\backslash(z \vee w)$. Then by residuation $x, y \leqslant x \vee y \leqslant (z \vee w)/u$, and by $(\vee/)$ we have $x \leqslant z/u \vee w/u$ and $y \leqslant z/u \vee w/u$. Observe that $x = x \wedge (z/u \vee w/u)$ and $y = y \wedge (z/u \vee w/u)$, and by distributivity we obtain that $x = x_1 \vee x_2$ and $y = y_1 \vee y_2$, where

$$x_1 = x \wedge (z/u),$$

$$x_2 = x \wedge (w/u),$$

$$y_1 = y \wedge (z/u),$$

$$y_2 = y \wedge (w/u).$$

Note that

$$x_1 \leqslant z/u \implies u \leqslant x_1\backslash z \leqslant (x_1 \wedge y_2)\backslash z,$$

$$x_2 \leqslant w/u \implies u \leqslant x_2\backslash w \leqslant (x_2 \wedge y_1)\backslash w,$$

$$y_1 \leqslant z/u \implies u \leqslant y_1\backslash z \leqslant (x_2 \wedge y_1)\backslash z,$$

$$y_2 \leqslant w/u \implies u \leqslant y_2\backslash w \leqslant (x_1 \wedge y_2)\backslash w.$$

Hence $u \leqslant (x_1 \wedge y_2)\backslash(z \wedge w) \leqslant x_1\backslash z \vee y_2\backslash w$ and $u \leqslant (x_2 \wedge y_1)\backslash(z \wedge w) \leqslant x_2\backslash z \vee y_1\backslash w$. Also, $u \leqslant x_1\backslash z \leqslant x_1\backslash z \vee y_1\backslash w$ and $u \leqslant y_2\backslash w \leqslant x_2\backslash z \vee y_2\backslash w$. This implies that:

$$\begin{aligned} u &\leqslant (x_1\backslash z \vee y_2\backslash w) \wedge (x_2\backslash z \vee y_1\backslash w) \wedge (x_1\backslash z \vee y_1\backslash w) \wedge (x_2\backslash z \vee y_2\backslash w) \\ &= ((x_2\backslash z \wedge x_1\backslash z) \vee y_1\backslash w) \wedge ((x_1\backslash z \wedge x_2\backslash z) \vee y_2\backslash w) \\ &= (x_1\backslash z \wedge x_2\backslash z) \vee (y_1\backslash w \wedge y_2\backslash w) \\ &= (x_1 \vee x_2)\backslash z \vee (y_1 \vee y_2)\backslash w \\ &= x\backslash z \vee y\backslash w. \end{aligned}$$

This proves the claim. □

Along the same lines, we obtain the following.

Proposition 2.1.6. *Let* **A** *be a distributive residuated binar.*

- *If* **A** *satisfies both* $(\backslash\vee)$ *and* $(/\wedge)$*, then* **A** *also satisfies* $(\vee/)$*.*
- *If* **A** *satisfies both* $(\cdot\wedge)$ *and* $(\vee/)$*, then* **A** *also satisfies* $(/\wedge)$*.*
- *If* **A** *satisfies both* $(\wedge\cdot)$ *and* $(\backslash\vee)$*, then* **A** *also satisfies* $(\wedge\backslash)$*.*

- *If* $\mathbf{A}$ *satisfies both* $(\wedge\backslash)$ *and* $(\cdot\wedge)$*, then* $\mathbf{A}$ *also satisfies* $(\wedge\cdot)$*.*

- *If* $\mathbf{A}$ *satisfies both* $(/\wedge)$ *and* $(\wedge\cdot)$*, then* $\mathbf{A}$ *also satisfies* $(\cdot\wedge)$*.*

Remark 2.1.7. The previous results were originally proven by passing to equivalent first-order conditions on dual structures via the Ackermann Lemma based algorithm (ALBA) (see, e.g., [15]). This foreshadows the utility of the duality theory discussed in Chapter 3. However, we shall not take a detour into first-order correspondence theory here.

Proposition 2.1.8. *Propositions 2.1.5 and 2.1.6 give the only implications among the six nontrivial distributive laws.*

Proof. We define residuated binars $\mathbf{A}_i$ for $i \in \{1,2,3,4,5,6\}$, each of whose lattice reduct is the four-element Boolean algebra $\{\bot, a, b, \top\}$, where $\bot < a, b < \top$. Tables for $\cdot, \backslash, /$ are given below. For $\mathbf{A}_1$:

$\cdot$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\top$

$\backslash$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	$\top$	$\top$	$\top$	$\top$
b	b	b	b	$\top$
$\top$	b	b	b	$\top$

$/$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	b	b
a	$\top$	$\top$	b	b
b	$\top$	$\top$	b	b
$\top$	$\top$	$\top$	$\top$	$\top$

For $\mathbf{A}_2$:

$\cdot$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	a	b	$\top$
$\top$	$\bot$	a	b	$\top$

$\backslash$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	$\top$	$\top$	$\top$	$\top$
b	$\bot$	a	b	$\top$
$\top$	$\bot$	a	b	$\top$

$/$	$\bot$	a	b	$\top$
$\bot$	$\top$	a	a	a
a	$\top$	$\top$	a	a
b	$\top$	a	$\top$	a
$\top$	$\top$	$\top$	$\top$	$\top$

For $\mathbf{A}_3$:

$\cdot$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	a	a
b	$\bot$	$\bot$	b	b
$\top$	$\bot$	$\bot$	$\top$	$\top$

$\backslash$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	a	$\top$	a	$\top$
b	a	a	$\top$	$\top$
$\top$	a	a	a	$\top$

$/$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\bot$
a	$\top$	$\top$	a	a
b	$\top$	$\top$	b	b
$\top$	$\top$	$\top$	$\top$	$\top$

For $\mathbf{A}_4$:

$\cdot$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a
b	$\bot$	a	$\bot$	a
$\top$	$\bot$	a	$\bot$	a

$\backslash$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	b	$\top$	b	$\top$
b	b	$\top$	b	$\top$
$\top$	b	$\top$	b	$\top$

$/$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\bot$
a	$\top$	$\top$	$\top$	$\top$
b	$\top$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$

For $\mathbf{A}_5$:

$\cdot$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	a
b	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	a	a	a

$\backslash$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	$\top$	$\bot$	$\top$
b	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\top$

$/$	$\bot$	a	b	$\top$
$\bot$	$\top$	b	b	b
a	$\top$	$\top$	$\top$	$\top$
b	$\top$	b	b	b
$\top$	$\top$	$\top$	$\top$	$\top$

For $\mathbf{A}_6$:

$\cdot$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	b	b
b	$\bot$	b	$\bot$	b
$\top$	$\bot$	b	b	b

$\backslash$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	a	a	$\top$	$\top$
b	b	b	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\top$

$/$	$\bot$	a	b	$\top$
$\bot$	$\top$	a	b	$\bot$
a	$\top$	a	b	$\bot$
b	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

One may check by direct computation that

- $\mathbf{A}_1 \models (/\wedge), (\wedge\backslash), (\wedge\cdot), (\cdot\wedge)$ and $\mathbf{A}_1 \not\models (\backslash\vee), (\vee/)$.
- $\mathbf{A}_2 \models (\backslash\vee), (\wedge\backslash), (\wedge\cdot), (\cdot\wedge)$ and $\mathbf{A}_2 \not\models (\vee/), (/\wedge)$.
- $\mathbf{A}_3 \models (\vee/), (/\wedge), (\wedge\cdot), (\cdot\wedge)$ and $\mathbf{A}_3 \not\models (\backslash\vee), (\wedge\backslash)$.
- $\mathbf{A}_4 \models (\vee/), (\backslash\vee), (/\wedge), (\cdot\wedge)$ and $\mathbf{A}_4 \not\models (\wedge\backslash), (\wedge\cdot)$.
- $\mathbf{A}_5 \models (\vee/), (\backslash\vee), (\wedge\backslash), (\wedge\cdot)$ and $\mathbf{A}_5 \not\models (/\wedge), (\cdot\wedge)$.
- $\mathbf{A}_6 \models (\vee/), (\backslash\vee), (/\wedge), (\wedge\backslash)$ and $\mathbf{A}_6 \not\models (\cdot\wedge), (\wedge\cdot)$.

For each $\sigma \in \{(\vee/), (\backslash\vee), (/\wedge), (\wedge\backslash), (\wedge\cdot), (\cdot\wedge)\}$, there is a unique implication in Proposition 2.1.5 or 2.1.6 having σ as its consequent. Let σ_1, σ_2 be equations in the antecedent of the aforementioned implication. Then the above countermodels show that if $\sigma \notin \Sigma \subseteq \{(\vee/), (\backslash\vee), (/\wedge), (\wedge\backslash), (\wedge\cdot), (\cdot\wedge)\}$ and either $\sigma_1 \notin \Sigma$ or $\sigma_2 \notin \Sigma$, then σ is not an equational consequence of Σ. This suffices to prove the claim. $\square$

The presence of complements and a neutral element in a residuated binar can have a profound impact on whether it satisfies any of the six nontrivial distributive laws, a stark example of which is illustrated by the following lemma.

Lemma 2.1.9. *Let* $\mathbf{A}$ *be a complemented residuated binar with neutral element* e*. If* $\mathbf{A}$ *is integral, then* $\wedge$ *and* $\cdot$ *coincide.*

Proof. From the fact that $\mathbf{A}$ is integral, we have that $x \cdot y \leqslant x \wedge y$ for all $x, y \in A$. Consequently, for any $x \in A$ we have that $x \cdot x' \leqslant x \wedge x' = \bot$, where x' denotes a complement of x. On the other hand, because the neutral element e is the top element of $\mathbf{A}$ we also have that $x \vee x' = e$ for any $x \in A$. Multiplying by x and using $(\cdot\vee)$, we obtain $x = x \cdot e = x \cdot (x \vee x') = x^2 \vee x \cdot x' = x^2 \vee \bot = x^2$. It follows that $\mathbf{A}$ is idempotent, whence for any $x, y \in A$, $x \wedge y = (x \wedge y) \cdot (x \wedge y) \leqslant x \cdot y \leqslant x \wedge y$, i.e., $x \cdot y = x \wedge y$. $\square$

The above entails that the only complemented integral residuated binars are Boolean algebras, and hence satisfy all six nontrivial distributive laws as well as lattice distributivity. Moreover, it turns out that the satisfaction of non-trivial distribution laws also often forces integrality in this setting.

Lemma 2.1.10. *Let* $\mathbf{A}$ *be a residuated binar with neutral element* e*. If* e *has a complement* e' *and* $\mathbf{A}$ *satisfies any one of the distributive laws* $(\cdot\wedge)$, $(\wedge\cdot)$, $(\wedge\backslash)$, $(/\wedge)$*, then* $\mathbf{A}$ *is integral.*

Proof. We prove the result for $(\cdot\wedge)$ and $(\wedge\backslash)$. The result will follow for $(\wedge\cdot)$ and $(/\wedge)$, respectively, by an entirely symmetric argument.

First, suppose that $\mathbf{A}$ satisfies $(\cdot\wedge)$. Note that

$$\begin{aligned} e' &= e \cdot e' \\ &\leqslant \top \cdot e' \\ &= \top \cdot e' \wedge \top \\ &= \top \cdot (e' \wedge e) \\ &= \top \cdot \bot \\ &= \bot. \end{aligned}$$

Thus $e' = \bot$, whence $e = \top$.

Second, suppose that $\mathbf{A}$ satisfies $(\wedge\backslash)$. Observe that

$$\begin{aligned} \top &= \bot\backslash\bot \\ &= (e \wedge e')\backslash\bot \\ &= (e\backslash\bot) \vee (e'\backslash\bot) \\ &= \bot \vee (e'\backslash\bot) \\ &= e'\backslash\bot, \end{aligned}$$

from which it follows that $\top \leqslant e'\backslash\bot$, and by residuation $e' \cdot \top \leqslant \bot$. Since $e \leqslant \top$ and $\cdot$ is order-preserving, we thus have $e' \cdot e \leqslant e' \cdot \top \leqslant \bot$. Therefore $e' \leqslant \bot$, i.e., $e' = \bot$. It follows as before that $e = e \vee \bot = e \vee e' = \top$, and this gives the result. □

Combining the previous two lemmas yields the following.

Corollary 2.1.11. *Let* $\mathbf{A}$ *be a complemented residuated binar with neutral element* e. *If* $\mathbf{A}$ *satisfies any one of the distributive laws* $(\cdot\wedge)$, $(\wedge\cdot)$, $(\wedge\backslash)$, $(/\wedge)$, *then* $\mathbf{A}$ *is a Boolean algebra.*

Proof. Because **A** is complemented, e has a complement. Lemma 2.1.10 then provides that **A** is integral, and hence from Lemma 2.1.9 it follows that **A** is a Boolean algebra. $\square$

Lemma 2.1.12. *Let* **A** *be a Boolean residuated binar with neutral element* e*, whose complement is denoted by* e'*. If* **A** *satisfies any one of the distributive laws* $(\cdot\wedge)$*,* $(\wedge\cdot)$*,* $(\backslash\vee)$*,* $(\vee/)$*,* $(\wedge\backslash)$*, or* $(/\wedge)$*, then* **A** *is integral, and hence is a Boolean algebra.*

Proof. Corollary 2.1.11 settles the claim if **A** satisfies any of $(\cdot\wedge)$, $(\wedge\cdot)$, $(\wedge\backslash)$, or $(/\wedge)$. We therefore prove the claim for **A** satisfying $(\backslash\vee)$; it will follow if **A** satisfies $(\vee/)$ by a symmetric argument. Suppose that **A** satisfies $(\backslash\vee)$. We have:

$$\begin{aligned}
\top &= \top\backslash\top \\
&= \top\backslash(e \vee e') \\
&= \top\backslash e \vee \top\backslash e' \\
&\leqslant \top\backslash e \vee e'
\end{aligned}$$

From the fact that Boolean algebras are $\wedge$-residuated, we obtain from the above that $\top = \top \wedge e \leqslant \top\backslash e$. Then from the residuation property for $\cdot$, we get $\top \leqslant e$. The result follows. $\square$

Corollary 2.1.13. *Let* **A** *be a Boolean residuated binar with a multiplicative neutral element. Then each of the identities* $(\cdot\wedge)$*,* $(\wedge\cdot)$*,* $(\backslash\vee)$*,* $(\vee/)$*,* $(\wedge\backslash)$*, and* $(\wedge\backslash)$ *is logically-equivalent to the other five.*

2.2 Lattices with involution

A *lattice with involution* (or *i-lattice* for short) is an algebra $\mathbf{A} = (A, \wedge, \vee, \neg)$, where $(A, \wedge, \vee)$ is a lattice and $\neg$ is an anti-isomorphism, i.e., an isomorphism of

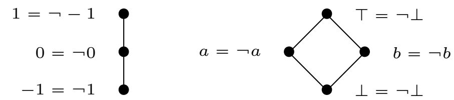


Figure 2.1: Labeled Hasse diagrams for $\mathbf{D}_3$ and $\mathbf{D}_4$

$(A, \wedge, \vee)$ and $(A, \vee, \wedge)$. Note that the latter requirement may be met equationally by stipulating that the identities

$$\neg(x \vee y) = \neg x \wedge \neg y,$$

$$\neg(x \wedge y) = \neg x \vee \neg y,$$

$$\neg\neg x = x$$

hold in $\mathbf{A}$, whence i-lattices form a variety. If $\mathbf{A}$ is an i-lattice, then $x \in A$ is called a *zero* if $\neg x = x$. We call an i-lattice distributive (modular) if its lattice reduct is distributive (modular), and we call it *normal*[3] if it satisfies the identity

$$x \wedge \neg x \leqslant y \vee \neg y \tag{N}$$

We will call expansions of normal distributive i-lattices by lattice bounds *Kleene algebras*. Observe that if $\bot$ and $\top$ are the least and greatest element of a Kleene algebra, then $\neg\bot = \top$ and $\neg\top = \bot$.

There are just three subdirectly irreducible distributive i-lattices: The two-element Boolean algebra with its usually involution; the three-element i-lattice chain $\mathbf{D}_3$; and the four-element i-lattice $\mathbf{D}_4$ with two incomparable zeros. Kalman showed in [41] that the variety of all distributive i-lattices is $\mathbb{ISP}(\mathbf{D}_4)$, and that the variety

[3]Note that our terminology differs from that introduced by Kalman [41]. In Kalman's terms, a normal i-lattice is one satisfying the given identity as well as distributivity.

of all normal distributive i-lattices is $\mathbb{ISP}(\mathbf{D}_3)$.[4] We denote the latter variety by NDIL, and the variety of Kleene algebras by KA.

2.2.1 Forbidden configurations

A lattice $\mathbf{A}$ is non-distributive if and only if neither of two forbidden sublattices appear in $\mathbf{A}$: The five-element non-modular lattice $\mathbf{N}_5$ and the five-element modular (but non-distributive) lattice $\mathbf{M}_3$. The forbidden configurations $\mathbf{N}_5$ and $\mathbf{M}_3$ provide a pictorial test for distributivity, and in this section we give an analogous test to determine whether a given modular i-lattice is normal.

Note that an i-lattice may have any number of zeros or no zero at all, but [41] shows that a modular i-lattice with a zero is normal if and only if the zero is unique. In light of this, we easily obtain the following.

Lemma 2.2.1. *Let $\mathbf{A}$ be a modular i-lattice with a zero. Then $\mathbf{A}$ refutes (N) if and only if $\mathbf{D}_4$ embeds into $\mathbf{A}$.*

Proof. Suppose first that $\mathbf{D}_4$ embeds into $\mathbf{A}$, and let a and b be the incomparable zeros of $\mathbf{D}_4$. Then $\neg a \wedge a = a \not\leq b = b \vee \neg b$, showing that $\mathbf{A}$ is not normal.

Conversely, suppose that $\mathbf{A}$ is not normal. Then $\mathbf{A}$ has two distinct zeros a and b by the above cited result of [41]. Note that distinct zeros are incomparable, whence a and b are incomparable. Then $\{a \wedge b, a, b, a \vee b\}$ is the universe of a subalgebra of $\mathbf{A}$ that is isomorphic to $\mathbf{D}_4$. $\square$

An i-lattice with no zeros may refute (N), and in this case $\mathbf{D}_4$ obviously does not appear as a subalgebra. Denote by $\mathbf{B}_8$ the i-lattice with no zeros whose lattice-reduct is the Boolean cube (see Figure 2.2). Our aim is to prove the following.

[4] Here and throughout the sequel we use $\mathbb{I}$, $\mathbb{H}$, $\mathbb{P}$, and $\mathbb{S}$ to denote the standard class operators of taking isomorphic copies, homomorphic images, direct products, and subalgebras.

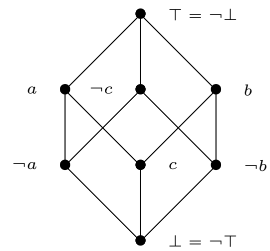


Figure 2.2: Labeled Hasse diagram for $\mathbf{B}_8$

Theorem 2.2.2. *Let* $\mathbf{A}$ *be a modular i-lattice with no zeros. Then* $\mathbf{A}$ *refutes* (N) *if and only if* $\mathbf{B}_8$ *embeds into* $\mathbf{A}$.

Toward this goal, we prove several technical lemmas.

Lemma 2.2.3. *Let* $\mathbf{A}$ *be an i-lattice with no zeros, and suppose that* $a, b \in A$ *with* $a \wedge \neg a \nleqslant b \vee \neg b$. *Then there exist* $a', b' \in A$ *with* $a' \wedge \neg a' \nleqslant b' \vee \neg b'$ *and* $\neg a' < a'$, $\neg b' < b'$.

Proof. Set $a' := a \vee \neg a$ and $b' := b \vee \neg b$. It is obvious that $\neg a' < a'$ and $\neg b' < b'$. Moreover, were it the case that $a' \wedge \neg a' \leqslant b' \vee \neg b'$, we would have $a \wedge \neg a \leqslant b \vee \neg b$, a contradiction. The result follows. □

Lemma 2.2.4. *Let* $\mathbf{A}$ *be a modular i-lattice with no zeros, and suppose that* $a, b \in A$ *with* $a \wedge \neg a \nleqslant b \vee \neg b$ *and* $\neg a < a$, $\neg b < b$. *Then:*

1. a *and* b *are incomparable.*

2. $\neg a$ *and* $\neg b$ *are incomparable.*

3. $\neg a$ *and* b *are incomparable.*

4. $a \wedge b \nleqslant \neg a \vee \neg b$.

Proof. The first three claims are trivial. For the fourth claim, suppose on the contrary that $a \wedge b \leqslant \neg a \vee \neg b$. Note that $a \wedge (\neg a \vee \neg b) \leqslant a \wedge (\neg a \vee b)$ because $\neg b \leqslant b$. On the other hand, observe that

$$\begin{aligned} a \wedge (b \vee \neg a) &= \neg a \vee (a \wedge b) \\ &\leqslant \neg a \vee (\neg a \vee \neg b) \\ &= \neg a \vee \neg b, \end{aligned}$$

whence it follows that $a \wedge (b \vee \neg a) = a \wedge (\neg a \vee \neg b)$. But notice that this implies that $\neg(a \wedge (b \vee \neg a)) = \neg a \vee (\neg b \wedge a) = a \wedge (\neg b \vee \neg a)$ by modularity, which contradicts the assumption that $\mathbf{A}$ has no zeros. $\square$

Lemma 2.2.5. *Let* $\mathbf{A}$ *be a modular i-lattice with no zeros, and suppose that* $a, b \in A$ *with* $a \wedge \neg a \nleqslant b \vee \neg b$ *and* $\neg a < a$, $\neg b < b$. *Then the elements* $a, b, \neg a, \neg b, a \wedge b, a \vee b, \neg a \wedge \neg b, \neg a \vee \neg b$ *are pairwise distinct.*

Proof. Note that a and b being incomparable, together with $\neg a < a$ and $\neg b < b$, gives that $a \neq b, \neg b, \neg a, a \wedge b, a \vee b, \neg a \wedge \neg b$. That $a \neq \neg a \vee \neg b$ follows because a and $\neg b$ are incomparable by Lemma 2.2.4. The same comments apply to b.

Were it the case that $\neg a = a \wedge b, a \vee b, \neg a \wedge \neg b$, or $\neg a \vee \neg b$, it would contradict the fact that $\neg a$ is incomparable to each of $a, b, \neg b$. The same holds for $\neg b$.

The above gives that each of $a, b, \neg a, \neg b$ is distinct from each of the remaining seven elements on the list. Lemma 2.2.4(4) gives that $a \wedge b \neq \neg a \vee \neg b$, and $a \wedge b < a < a \vee b$ since a and b are incomparable. Were $a \wedge b = \neg a \vee \neg b = \neg(a \wedge b)$, it would contradict the fact that $\mathbf{A}$ has no zeros. Similar comments show that $a \vee b$ is distinct from the remaining elements on the list. Finally, $\neg a \neq \neg b$ implies that $\neg a \wedge \neg b \neq \neg a \vee \neg b$. This proves the claim. $\square$

Lemma 2.2.6. *Let* $\mathbf{A}$ *be a modular i-lattice with no zeros, and suppose that* $a, b \in L$ *with* $a \wedge \neg a \nleqslant b \vee \neg b$ *and* $\neg a < a$, $\neg b < b$. *Further assume that*

$$a \leqslant \neg a \vee b$$

$$b \leqslant \neg b \vee a$$

Then $S = \{a, b, \neg a, \neg b, a \wedge b, a \vee b, \neg a \wedge \neg b, \neg a \vee \neg b\}$ *is the universe of a subalgebra of* $\mathbf{A}$.

Proof. That S is closed under $\neg$ follows from the De Morgan laws and the fact that $\neg\neg x = x$ for all $x \in L$. Because closure under $\neg$ and either of the lattice connectives implies closure under the other lattice connective, it suffices to show that S is closed under $\vee$. There are only seven cases when this is not obvious, and we check them in turn. Using modularity and $a \leqslant \neg a \vee b$, we have

$$a \vee b \leqslant \neg a \vee b \vee b = \neg a \vee b \leqslant a \vee b \implies \neg a \vee b = a \vee b$$

$$b \vee \neg a \vee \neg b = \neg a \vee b = a \vee b$$

$$\neg a \vee (b \wedge a) = (\neg a \vee b) \wedge a = (a \vee b) \wedge a = a$$

Using $b \leqslant \neg b \vee a$,

$$a \vee b \leqslant a \vee \neg b \vee a = a \vee \neg b \leqslant a \vee b \implies a \vee \neg b = a \vee b$$

$$a \vee \neg a \vee \neg b = a \vee \neg b = a \vee b$$

$$\neg b \vee (a \wedge b) = (\neg b \vee a) \wedge b = (a \vee b) \wedge b = b$$

$$(a \wedge b) \vee (\neg a \vee \neg b) = ((a \wedge b) \vee \neg a) \vee ((a \wedge b) \vee \neg b) = a \vee b$$

Because S is closed under each of the operations of $\mathbf{A}$, it follows that S is the universe of a subalgebra of $\mathbf{A}$. This proves the claim. $\square$

Lemma 2.2.7. *Let $\mathbf{A}$ be a modular i-lattice with no zeros, and suppose that $a, b \in A$ with $a \wedge \neg a \nleqslant b \vee \neg b$ and $\neg a < a$, $\neg b < b$. Set $a' := \neg a \vee (b \wedge a)$ and $b' := \neg b \vee (a \wedge b)$. Then $\neg a' < a'$, $\neg b' < b'$, $a' \wedge \neg a' \nleqslant b' \vee \neg b'$, $a' \leqslant \neg a' \vee b'$, and $b' \leqslant \neg b' \vee a'$. Hence $\mathbf{A}$ with a' and b' satisfy the hypotheses of Lemma 2.2.6.*

Proof. A direct calculation using modularity shows that $\neg a' \leqslant a'$ and $\neg b' \leqslant b'$, and these inequalities are strict because $\mathbf{A}$ has no zeros. Observe that

$$\begin{aligned}\neg a' \vee b' &= \neg a \vee (\neg b \wedge a) \vee \neg b \vee (a \wedge b) \\ &= (\neg a \vee \neg b) \vee (a \wedge b) \\ &\geqslant \neg a \vee (b \wedge a) \\ &= a'\end{aligned}$$

This shows that $a' \leqslant \neg a' \vee b'$, and by symmetry $b' \leqslant \neg b' \vee a'$.

For the rest, suppose toward a contradiction that $a' \wedge \neg a' \leqslant b' \vee \neg b'$, i.e., $\neg a' \leqslant b'$. By modularity, this amounts to $\neg a \vee (\neg b \wedge a) \leqslant b \wedge (a \vee \neg b)$. But this implies that $\neg a \leqslant b$, contradicting Lemma 2.2.4(3) and completing the proof. $\square$

Theorem 2.2.2 follows immediately from the foregoing lemma, and combining this with Lemma 2.2.1 we obtain the following.

Theorem 2.2.8. *Let $\mathbf{A}$ be a modular i-lattice. Then $\mathbf{A}$ is normal if and only if neither of the i-lattices $\mathbf{D}_4$ or $\mathbf{B}_8$ may be embedded in $\mathbf{A}$.*

2.3 Commutative residuated lattices and involutivity

The variety of residuated lattices is probably the most important class of residuated structures, and for our purposes certain expansions of commutative residuated lattices (henceforth CRLs) occupy an especially central role. For us, their importance arises because of their deep connection to several nonclassical logics (especially relevant and many-valued logics), for which they provide the *equivalent algebraic semantics* in the sense of [5]. We shall not dwell on the details of this connection here, but refer the reader to [29, Section 2.6] for details.

In addition to its logical importance, the variety CRL of CRLs also enjoys numerous pleasant algebraic properties: It is an arithmetical variety with the congruence extension property, and each congruence of a CRL is determined by the congruence class of its multiplicative identity. The following gives some useful properties of CRL, all of which are well-known in the literature (and many of which rephrase facts from the general setting of residuated binars).

Proposition 2.3.1. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e)$ *be a CRL. Then* $\mathbf{A}$ *satisfies the following.*

1. $x(x \to y) \leqslant y$.
2. $x(y \vee z) = xy \vee xz$.
3. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.
4. $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.
5. $(xy) \to z = x \to (y \to z) = y \to (x \to z)$.
6. $e \to x = x$.
7. $e \leqslant x \to x$.

We introduce several subvarieties of CRL that will be important later. Note first that a distributive CRL $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, e)$ is semilinear precisely when it satisfies the identity

$$e \leqslant (x \rightarrow y) \vee (y \rightarrow x),$$

and therefore the semilinear members of CRL form a variety in their own right. The integral semilinear CRLs are called *generalized monoidal t-norm logic algebras*, or *GMTL*-algebras for short, due to the fact that they provide the equivalent algebraic semantics for the negation-free fragment of Esteva and Godo's monoidal t-norm based logic [22]. Some authors also call GMTL-algebras *prelinear semihoops*. Bounded GMTL-algebras are called *monoidal t-norm logic algebras*, or MTL-algebras, and provide the equivalent algebraic semantics for monoidal t-norm based logic (with negation). Note that for bounded integral CRLs, we usually use 1 for the multiplicative neutral element (which is also the greatest element), and 0 for the least element. We denote the varieties of GMTL-algebras and MTL-algebras by GMTL and MTL, respectively.

Note that in an MTL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, 1, 0)$, it is common practice to define additional operations $\neg$ and $+$ on A by

$$\neg x := x \rightarrow 0 \text{ and } x + y := \neg(\neg x \cdot \neg y).$$

So defined, $+$ is a commutative operation. Moreover, $\neg$ satisfies the the De Morgan laws due to the identities $(\wedge\backslash)$ and $(\vee\backslash)$, but may not satisfy the law of double negation $\neg\neg x = x$. An MTL-algebra that satisfies the latter condition is called *involutive*.

An MTL-algebra $\mathbf{A}$ is said to *have no zero divisors* if for all $x, y \in A$, $x \cdot y = 0$ implies $x = 0$ or $y = 0$. An MTL-algebra is called an *SMTL-algebra* if it satisfies the identity $x \wedge \neg x = 0$. The subvariety of MTL consisting of the SMTL-algebras is denoted by SMTL.

The following appears in [44, Proposition 4.14] in the context of totally-ordered algebras, but comes from [27] in its full generality.

Proposition 2.3.2. *Let* $\mathbf{A}$ *be an MTL-algebra. Then* $\mathbf{A}$ *has no zero divisors if and only if* $\mathbf{A}$ *is a directly-indecomposable SMTL-algebra.*

Proof. Suppose that $\mathbf{A}$ has no zero divisors. If $x \in A$, then $x \cdot \neg x = x \cdot (x \to 0) = 0$, giving $x = 0$ or $\neg x = x \to 0 = 0$ by the hypothesis. If either $x = 0$ or $\neg x = 0$, then $x \wedge \neg x = 0$ as well, and thus $\mathbf{A}$ is an SMTL-algebra. If $\mathbf{A}$ may be written as a direct product $\mathbf{A}_1 \times \mathbf{A}_2$ of nontrivial MTL-algebras, then we have $(1,0) \cdot (0,1) = (0,0)$ although $(1,0),(0,1)$ are nonzero. This contradicts $\mathbf{A}$ having no zero divisors, so $\mathbf{A}$ is directly indecomposable.

Conversely, if $\mathbf{A}$ is a directly-indecomposable SMTL-algebra, then $\mathbf{A}$ may be written as an ordinal sum of the form $\mathbf{2} \oplus \mathbf{B}$, where $\mathbf{2}$ is the two-element MTL-algebra and $\mathbf{B}$ is a GMTL-algebra (see, e.g., [1]). In this event, $x \cdot y = 0$ only if $x = 0$ or $y = 0$, completing the proof. $\square$

A CRL for which $\wedge$ coincides with $\cdot$ is called a *Brouwerian algebra*, and the bounded Brouwerian algebras are called *Heyting algebras*. We denote the varieties of Brouwerian algebras and Heyting algebras by BrA and HA, respectively. The semilinear Brouwerian algebras and Heyting algebras are called, respectively, *relative Stone algebras* and *Gödel algebras*, and by the above they form varieties that we denote by RSA and GA. Relative Stone algebras and Gödel algebras are examples of integral semilinear residuated structures, and are therefore subvarieties of GMTL and MTL, respectively. We will make use of the following special property of these varieties in the sequel.

Proposition 2.3.3 ([31, Lemma 4.1])**.** *Let* $\mathbf{A} \in \mathsf{RSA}$ *and let* $a, b \in A$*. Then the following are equivalent.*

1. $a \to b = b$ *and* $b \to a = a$.

2. $a \vee b = e$.

An expansion of a CRL $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e)$ by a unary operation $\neg$ is called an *involutive* CRL if it satisfies $\neg\neg x = x$ and $x \to \neg y = y \to \neg x$. It is easy to show that involutive CRLs satisfy the De Morgan laws $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$, and hence if $(A, \wedge, \vee, \cdot, \to, e, \neg)$ is an involutive CRL, then $(A, \wedge, \vee, \neg)$ is an i-lattice.

Note that an involutive CRL $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e, \neg)$ satisfies $\neg x = e \to \neg x = x \to \neg e$, whence the involution of an involutive CRL is definable in terms of the constant $f := \neg e$. It turns out that involutive CRLs are term-equivalent to expansions of CRLs by a constant f such that $x = (x \to f) \to f$ for all x, whence we may freely consider involutive CRLs as pointed CRLs. If $(A, \wedge, \vee, \cdot, \to, e, \neg)$ is an integral involutive CRL, then for arbitrary $x \in A$ we have $f = \neg e \leqslant x$ as a consequence of $\neg x \leqslant e$. Thus integral involutive CRLs are bounded with f being the least element. In particular, this means that our definition of involutive MTL-algebras above agrees with our definition of involutive CRLs.

The following ties together much of the material introduced in this chapter.

Proposition 2.3.4. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e, \neg)$ *be a semilinear involutive CRL. Then* $(A, \wedge, \vee, \neg)$ *is a normal distributive i-lattice.*

Proof. It suffices to check the claim on generating algebras, so suppose that $\mathbf{A}$ is a linearly-ordered involutive CRL and let $x, y \in A$. Then $x \leqslant y$ or $y \leqslant x$. If $x \leqslant y$, then $x \wedge \neg x \leqslant y \leqslant y \vee \neg y$. If $y \leqslant x$, then $\neg x \leqslant \neg y$ and hence $x \wedge \neg x \leqslant y \vee \neg y$. $\square$

Given a CRL $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e)$, a *nucleus* on $\mathbf{A}$ is a map $N\colon A \to A$ such that:

1. N is a closure operator on $(A, \wedge, \vee, \cdot, \to, e)$, i.e.
    (a) N is expanding ($x \leqslant Nx$ for all $x \in A$).
    (b) N is isotone (if $x, y \in A$ with $x \leqslant y$, then $Nx \leqslant Ny$).
    (c) N is idempotent ($NNx = Nx$ for all $x \in A$).
2. $Nx \cdot Ny \leqslant N(x \cdot y)$ for all $x, y \in A$.

If $\mathbf{A}$ is a CRL and N is a nucleus on $\mathbf{A}$, then the N-*nuclear image* of $\mathbf{A}$ is the algebra $\mathbf{A}_N = (N[A], \wedge, \vee_N, \cdot_N, \to, Ne)$, where for all $x, y \in A$,

$$x \vee_N y = N(x \vee y)$$

$$x \cdot_N y = N(x \cdot y)$$

Nuclear images of CRLs are again CRLs. The following gives an important example of nuclei that we will return to later.

Example 2.3.5. *Let $\mathbf{A} = (A, \wedge, \vee, \to, e)$ be a Brouwerian algebra. For each $d \in A$, the function $N\colon \mathbf{A} \to \mathbf{A}$ defined by $Na = d \to a$ is a nucleus on $\mathbf{A}$.*

2.3.1 srDL-algebras

An MTL-algebra is called an *srDL-algebra* if it satisfies the identities

$$\neg(x^2) \to (\neg\neg x) = 1 \text{ and } (x+x)^2 = x^2 + x^2.$$

The involutive srDL-algebras are called *sIDL-algebras*. The varieties of srDL-algebras and sIDL-algebras are respectively denoted by srDL and sIDL.[5]

A *deductive filter* of an srDL-algebra $\mathbf{A}$ is a lattice filter $\mathfrak{x}$ of $\mathbf{A}$ such that if $x, x \to y \in \mathfrak{x}$, then $y \in \mathfrak{x}$, and the *radical* $\mathscr{R}(\mathbf{A})$ of an srDL-algebra $\mathbf{A}$ is the intersection of $\mathbf{A}$'s maximal deductive filters. From [1, Proposition 2.5], the radical of $\mathbf{A}$ is exactly the set

$$\mathscr{R}(\mathbf{A}) = \{x \in A : \neg x < x\}.$$

For any srDL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 1, 0)$, $\mathscr{R}(\mathbf{A})$ is a subalgebra of the 0-free reduct of $\mathbf{A}$, and consequently it is a GMTL-algebra. If $\mathbf{A}$ is an srDL-algebra, then the coradical of $\mathbf{A}$ is

$$\mathscr{C}(\mathbf{A}) := \{x \in \mathbf{A} : \neg x \in \mathscr{R}(\mathbf{A})\}.$$

The *Boolean skeleton* of an srDL-algebra $\mathbf{A}$ is the largest subalgebra of $\mathbf{A}$ that is a Boolean algebra, and it is denoted by $\mathscr{B}(\mathbf{A})$. For an srDL-algebra $\mathbf{A}$, elements in $\mathscr{R}(\mathbf{A})$, $\mathscr{C}(\mathbf{A})$, and $\mathscr{B}(\mathbf{A})$ are respectively called *radical elements*, *coradical elements*, and *Boolean elements*.

The following two lemmas give information about these special subsets of an srDL-algebra.

Lemma 2.3.6. *[10, Lemma 1.5] Let* $\mathbf{A}$ *be an srDL-algebra. Then*

1. *If* $u \in \mathscr{B}(\mathbf{A})$, *then* $\neg u \in \mathscr{B}(\mathbf{A})$ *and* $\neg\neg u = u$.

2. *An element* $u \in A$ *is Boolean if and only if* $u \vee \neg u = 1$.

If $u \in \mathscr{B}(\mathbf{A})$ *and* $a, b \in A$, *then*

3. $u \cdot a = u \wedge a$,

4. $u \to a = \neg u \vee a$,

[5] These names are drawn from [55].

5. $a = (a \wedge u) \vee (a \wedge \neg u)$,

6. *If* $a \wedge b \geqslant \neg u$ *and* $u \wedge a = u \wedge b$, *then* $a = b$.

Lemma 2.3.7 ([1],[55])**.** *Let* $\mathbf{A}$ *be an srDL-algebra. Then:*

1. $\mathscr{C}(\mathbf{A}) = \{\neg x : x \in \mathscr{R}(\mathbf{A})\} = \{x \in A : x < \neg x\}$.

2. *For every* $y \in \mathscr{R}(\mathbf{A})$, $x \in \mathscr{C}(\mathbf{A})$, $x < y$.

3. *If* $\mathbf{A}$ *is directly-indecomposable, then* $\mathbf{A} \cong \mathscr{R}(\mathbf{A}) \cup \mathscr{C}(\mathbf{A})$.

In any srDL-algebra $\mathbf{A}$, there is a representation (see [1]) of each element of A in terms of $\mathscr{R}(\mathbf{A})$ and $\mathscr{B}(\mathbf{A})$. In particular, if $a \in A$ then there exist $x \in \mathscr{R}(\mathbf{A})$ and $u \in \mathscr{B}(\mathbf{A})$ so that

$$a = (u \vee \neg x) \wedge (\neg u \vee x) = (u \wedge x) \vee (\neg u \wedge \neg x). \tag{2.3.1}$$

In Chapter 8, we make extensive use of this representation when we work with srDL-algebras.

2.3.2 Sugihara monoids

A *Sugihara monoid* is a distributive, idempotent, involutive CRL. Sugihara monoids turn out to be semilinear [2], and consequently Proposition 2.3.4 provides that the $(\wedge, \vee, \neg)$-reduct of each Sugihara monoid lies in $\mathbb{ISP}(\mathbf{D}_3)$. This observation proves crucial to our development of a duality theory for Sugihara monoids in Chapter 6.

We provide several examples of Sugihara monoids, which we will return to in later chapters.

Example 2.3.8. *Let* $\mathbf{S} := (\mathbb{Z}, \wedge, \vee, \cdot, \rightarrow, 0, -)$, *where the lattice order is the usual order on the integers,* $-$ *is the additive inversion on the integers, and the multiplication* $\cdot$ *is given by:*

$$x \cdot y = \begin{cases} x & |x| > |y| \\ y & |x| < |y| \\ x \wedge y & |x| = |y| \end{cases}$$

The residual $\to$ *is given by:*

$$x \to y = \begin{cases} (-x) \vee y & x \leqslant y \\ (-x) \wedge y & x \nleqslant y \end{cases}$$

Then $\mathbf{S}$ *is a Sugihara monoid.*

A Sugihara monoid is called *odd* if it satisfies $\neg e = e$. The Sugihara monoid $\mathbf{S}$ given above is odd.

Example 2.3.9. *Let* $\mathbf{S}\backslash\{0\} := (\mathbb{Z}\backslash\{0\}, \wedge, \vee, \cdot, \to, 1, -)$*, where each of* $\wedge, \vee, \cdot, \to$*, and* $-$ *are as in Example 2.3.8. Then* $\mathbf{S}\backslash\{0\}$ *is a Sugihara monoid where the monoid identity is* 1*. Note that since* $\neg 1 = -1$*,* $\mathbf{S}\backslash\{0\}$ *is not odd.*

Example 2.3.10. *Given a positive integer* n*, we define a totally-ordered Sugihara monoid with* n *elements as follows. If* $n = 2m+1$ *is odd,* $\{-m, \ldots, -1, 0, 1, \ldots, m\}$ *is the universe of a subalgebra of* $\mathbf{S}$ *that has* n *elements. If* $n = 2m$ *is even, then the set* $\{-m, \ldots, -1, 1, \ldots m\}$ *is the universe of a subalgebra of* $\mathbf{S}\backslash\{0\}$ *that has* n *elements. In each case, the Sugihara monoid with* n *elements just defined will be denoted by* $\mathbf{S}_n$*. Note that* $\mathbf{S}_n$ *is an odd Sugihara monoid if and only if* n *is an odd integer.*

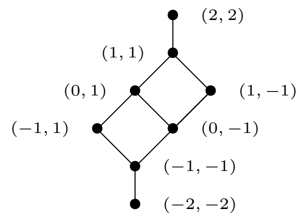


Figure 2.3: Labeled Hasse diagram for $\mathbf{E}$

Example 2.3.11. *In each of the previous examples, the Sugihara monoids defined are chains. We give a nonlinear example as follows. Consider the set*

$$E = \{(-2,-2), (-1,-1), (-1,1), (0,-1), (0,1), (1,-1), (1,1), (2,2)\}.$$

Then E forms the universe of a subalgebra of $\mathbf{S}_5 \times \mathbf{S}_4$. Figure 2.3 depicts the Hasse diagram for $\mathbf{E}$. We will use $\mathbf{E}$ to illustrate our work on Sugihara monoids in later chapters.

We conclude our preliminary discussion of Sugihara monoids with the following proposition, which shows the special role of the examples $\mathbf{S}$ and $\mathbf{S}\backslash\{0\}$ in the theory of Sugihara monoids (see, e.g., [48]).

Proposition 2.3.12. *The Sugihara monoids are generated as a quasivariety by $\{\mathbf{S}, \mathbf{S}\backslash\{0\}\}$.*

We denote the variety of Sugihara monoids by SM and the variety of odd Sugihara monoids by OSM. Their varieties consisting of their bounded expansions will be denoted by $\mathsf{SM}_\perp$ and $\mathsf{OSM}_\perp$.

Remark 2.3.13. Note that whenever K is a class of similar algebras, we freely consider K as a category whose objects are algebras in K and whose morphisms are algebraic homomorphisms (in the appropriate similarity type) between them. In particular, we consider varieties and quasivarieties as categories in this fashion. We thus use NDIL, KA, CRL, GMTL, MTL, srDL, BrA, HA, RSA, GA, SM, OSM, $\mathsf{SM}_\perp$, and $\mathsf{OSM}_\perp$ to denote the categories of algebras in each given class as well as the varieties.

Chapter 3

Duality theory

Having introduced in Chapter 2 the algebraic structures we are concerned with, we turn to a discussion of our chief tool for their study: Topological dualities for lattice-based algebras. Duality theory has its origin in Stone's representation theorem for Boolean algebras [53], and has been extended to distributive lattices [49, 50], Heyting algebras [21], and expansions of these algebras by operators [39, 40, 35].

Duality theory is the subject of a vast literature. For background on natural duality theory, we refer to [14]. For information on Stone duality we refer to [38], and for the duality theory of Boolean algebras with operators, we refer to [34].

Most of this chapter introduces preliminary material, but the extension of the Davey-Werner duality to normal distributive i-lattices (see Section 3.3) was developed in the author's [24], and the duality for GMTL-algebras (see Section 3.4.1) descends from the author's [27].

3.1 Natural dualities

Natural duality theory gives one of the most general and highly-developed frameworks available for discussing topological dualities. In addition to providing context for the classical dualities discussed in Section 3.2, natural duality theory is necessary in Section 3.3 to obtain some preliminary results toward our duality for Sugihara monoids in Chapter 6. Our treatment in this section is essentially drawn from [14].

Suppose that $\underline{\mathbf{M}}$ is a finite algebra, and set $\mathsf{A} := \mathbb{ISP}(\underline{\mathbf{M}})$. We consider an *enriched topological space* $\underset{\sim}{\mathbf{M}} = (M, G, H, R, \tau)$ defined on the same carrier M as $\underline{\mathbf{M}}$, where

- G is a set of total operations on M,
- H is a set of partial operations on M,
- R is a set of relations on M, and
- τ is the discrete topology on M.

Define a category S such that:

- The objects of S are enriched topological spaces in $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$, the class of isomorphic copies of topologically-closed subspaces of nonempty powers of $\underset{\sim}{\mathbf{M}}$.
- The morphisms of S are continuous homomorphisms between members of $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$.

Observe that the graph of each element of $G \cup H$, as well as each element of R, may be considered as a subset of some direct power $\underline{\mathbf{M}}$, and when each of these subsets is a subalgebra of the appropriate of power of $\underline{\mathbf{M}}$ we say that $\underset{\sim}{\mathbf{M}}$ is *algebraic* over $\underline{\mathbf{M}}$. When $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, there is an adjunction between A and S. The functors $\mathcal{D}\colon \mathsf{A} \to \mathsf{S}$ and $\mathcal{E}\colon \mathsf{S} \to \mathsf{A}$ of this adjunction are defined on objects by

$$\mathcal{D}(\mathbf{A}) = \mathrm{Hom}_{\mathsf{A}}(\mathbf{A}, \underline{\mathbf{M}}),$$

$$\mathcal{E}(\mathbf{X}) = \mathrm{Hom}_{\mathsf{S}}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$$

where $\mathrm{Hom}_{\mathsf{A}}(\mathbf{A}, \underline{\mathbf{M}})$ inherits its structure pointwise from $\underset{\sim}{\mathbf{M}}$, and $\mathrm{Hom}_{\mathsf{S}}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ inherits its structure pointwise from $\underline{\mathbf{M}}$. For morphisms $h\colon \mathbf{A} \to \mathbf{B}$ in A and $\alpha\colon \mathbf{X} \to \mathbf{Y}$ in S, $\mathcal{D}(h)\colon \mathcal{D}(\mathbf{B}) \to \mathcal{D}(\mathbf{A})$ and $\mathcal{E}(\alpha)\colon \mathcal{E}(\mathbf{Y}) \to \mathcal{E}(\mathbf{X})$ are defined by

$$\mathcal{D}(h)(x) = x \circ h$$

$$\mathcal{E}(\alpha)(x) = x \circ \alpha,$$

respectively. The unit of this adjunction is the natural transformation e given by evaluation, i.e., for objects $\mathbf{A}$ of $\mathcal{A}$, $e_{\mathbf{A}}\colon \mathbf{A} \to \mathcal{E}\mathcal{D}(\mathbf{A})$ is defined by $e_{\mathbf{A}}(a)(x) = x(a)$. The counit is likewise defined for objects $\mathbf{X}$ of S by $\epsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathcal{D}\mathcal{E}(\mathbf{X})$ given by $\epsilon_{\mathbf{X}}(x)(\alpha) = \alpha(x)$. With the above set-up, whenever each homomorphism $e_{\mathbf{A}}$ is an isomorphism, we say that the dual adjunction $(\mathcal{D}, \mathcal{E}, e, \epsilon)$ is a *natural duality.* We also say that the structure $\underset{\sim}{\mathbf{M}}$ *dualizes* $\underline{\mathbf{M}}$. When each $\epsilon_{\mathbf{X}}$ is also an isomorphism, we say that the natural duality $(\mathcal{D}, \mathcal{E}, e, \epsilon)$ is *full.* A duality is full precisely when it is a dual equivalence between the categories A and S. When a natural duality $(\mathcal{D}, \mathcal{E}, e, \epsilon)$ associates embeddings in S with surjections in A (equivalently, embeddings in A with with surjections in S) we say that the duality is *strong.* Strong dualities are full, but the converse is not in general true.

Suppose that $\underset{\sim}{\mathbf{M}} = (M, G, H, R, \tau)$ and $\underset{\sim}{\mathbf{M}}' = (M, G', H', R', \tau)$ are discrete topological structures that dualize the same finite algebra $\underline{\mathbf{M}}$, and let s be an algebraic relation, operation, or term on $\underline{\mathbf{M}}$. We say that $\underset{\sim}{\mathbf{M}}$ (or $G \cup H \cup R$) *entails* s on $\mathcal{D}(\mathbf{A})$ if every continuous map $\alpha\colon \mathcal{D}(\mathbf{A}) \to M$ preserving the all relations, operations, and partial operations in $G \cup H \cup R$ also preserves s. We say that $G \cup H \cup R$ *entails* s if $G \cup H \cup R$ entails s on $\mathcal{D}(\mathbf{A})$ for every $A \in \mathcal{A}$. If $G \cup H \cup R$ entails s for every $s \in G' \cup H' \cup R'$ we say that $G \cup H \cup R$ entails $G' \cup H' \cup R'$ or that $\underset{\sim}{\mathbf{M}}$ entails $\underset{\sim}{\mathbf{M}}'$. If P is a set of (partial and total) operations on M of finite arity, $\mathbf{A}$ is a subalgebra of $\underline{\mathbf{M}}^S$ for some (not necessarily finite) set S, and $h\colon A \to M$ is an algebraic (partial or total) operation on $\underline{\mathbf{M}}$, then we say that P *hom-entails* h if every subset of a power of M which is closed under the operations in P is closed under h. We say that $G \cup H \cup R$ *strongly entails* $G' \cup H' \cup R'$ if $G \cup H \cup R$ entails $G' \cup H' \cup R'$ and $G \cup H$ hom-entails every operation in $G' \cup H'$.

For a finite algebra $\mathbf{A}$, define $\mathrm{irr}(\mathbf{A})$ to be the least $n \in \omega$ such that the diagonal congruence Δ is the meet of n meet-irreducible congruences in the congruence lattice of $\mathbf{A}$. We define the *irreducibility index* of a finite algebra $\underline{\mathbf{M}}$ to be $\mathrm{Irr}(\underline{\mathbf{M}}) = \mathsf{max}\{\mathrm{irr}(\mathbf{A}) : \mathbf{A} \leqslant \underline{\mathbf{M}}\}$. Also denote by K the set of one-element subalgebras of $\underline{\mathbf{M}}$, $\mathcal{B}_n$ the set of all n-ary relations algebraic over $\underline{\mathbf{M}}$, and $\mathcal{P}_n$ the set of all n-ary partial operations algebraic over $\underline{\mathbf{M}}$. One of the fundamental tools for producing strong dualities for the prevariety generated by a finite algebra with a near-unanimity term is the following NU strong duality theorem.

Theorem 3.1.1 ([14], Theorem 3.3.8)**.** *Let $k \geqslant 2$ and assume that $\underline{\mathbf{M}}$ has a $(k+1)$-ary near-unanimity term. If*

$$\underset{\sim}{\mathbf{M}} = (M, K, H, \mathcal{B}_k, \tau)$$

where

$$H = \bigcup\{\mathcal{P}_n : 1 \leqslant n \leqslant \mathrm{Irr}(\underline{\mathbf{M}})\}$$

then any structure that strongly entails $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\underline{\mathbf{M}}$.

Since algebras with a lattice reduct always have a majority term, the above theorem may be applied to lattice-based algebras to obtain the following.

Corollary 3.1.2 ([14], Corollary 3.3.9)**.** *Suppose that $\underline{\mathbf{M}}$ is a finite algebra with a lattice reduct, and that all the non-trivial subalgebras of $\underline{\mathbf{M}}$ are subdirectly irreducible.*

Then any structure that strongly entails $\underset{\sim}{\mathbf{M}} = (M, K, \mathcal{P}_1, \mathcal{B}_2, \tau)$ *yields a strong duality on* $\underline{\mathbf{M}}$.

The following $\underset{\sim}{\mathbf{M}}$-Shift Strong Duality Lemma underwrites the applications of (strong) entailment to follow.

Theorem 3.1.3 ([14], Lemma 3.2.3)**.** *Consider the structure* $\underset{\sim}{\mathbf{M}}' = (M, G', H', R', \tau)$.

1. *If* $\underset{\sim}{\mathbf{M}}$ *strongly entails* $\underset{\sim}{\mathbf{M}}'$ *and* $\underset{\sim}{\mathbf{M}}'$ *yields a strong duality on* $\mathcal{A}$*, then* $\underset{\sim}{\mathbf{M}}$ *also yields a strong duality on* $\mathcal{A}$.
2. $\underset{\sim}{\mathbf{M}}$ *strongly entails* $\underset{\sim}{\mathbf{M}}'$ *if it is obtained from* $\underset{\sim}{\mathbf{M}}'$ *by*
   (a) *enlarging* G', H', *or* R',
   (b) *deleting members of* G' *or* H' *which can be obtained as compositions of the remaining members of* G' *and* H' *and the projection mappings, or*
   (c) *deleting a member* h *of* H' *which has an extension among the remaining members of* $G' \cup H'$ *and adding* $\mathrm{dom}(h)$ *to* R'.
3. $\underset{\sim}{\mathbf{M}}$ *strongly entails* $\underset{\sim}{\mathbf{M}}'$ *if* $\underset{\sim}{\mathbf{M}}$ *entails* $\underset{\sim}{\mathbf{M}}'$ *and is obtained from* $\underset{\sim}{\mathbf{M}}'$ *by*
   (a) *deleting members of* R'*, or*
   (b) *deleting members of* H' *which have an extension in* G' *or* H'.

Although the preceding results give a method for producing a category dual to $\mathbb{ISP}(\underline{\mathbf{M}})$ for many finite algebras $\underline{\mathbf{M}}$, the dual category $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is not especially transparent. The final two results of this section provide a method for finding a more user-friendly description of the members of $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. Given a first-order language $\mathcal{L}$, recall that the *quasiatomic formulas* of $\mathcal{L}$ consist of the atomic formulas of $\mathcal{L}$, the negated atomic formulas of $\mathcal{L}$, and the expressions of the form

$$\bigwedge_{i=1}^{n} \alpha_i \Rightarrow \alpha_{n+1},$$

where $n \geqslant 1$ and α_i is an atomic formula of $\mathcal{L}$ for each $i \in \{1, \dots, n+1\}$. The next two results are often called the *preservation* and *separation* theorems. We delay examples of how the foregoing machinery may be used until after the next section.

Theorem 3.1.4 ([14], Theorem 1.4.3)**.** *Let $\underset{\sim}{\mathbf{M}}$ be a finite, discrete structured topological space and let $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$.*

1. *$\mathbf{X}$ is a structured topological space which satisfies every quasiatomic formula that is satisfied by $\underset{\sim}{\mathbf{M}}$, and as a topological space $\mathbf{X}$ is a compact Hausdorff space with a basis of clopen sets.*

2. *If h is an n-ary function or partial function symbol, then the domain of $h^{\mathbf{X}}$ is is a closed subset of $\mathbf{X}^n$ and $h^{\mathbf{X}}$ is continuous.*

3. *If r is an n-ary relation symbol, then $r^{\mathbf{X}}$ is a closed subset of $\mathbf{X}^n$.*

Theorem 3.1.5 ([14], Theorem 1.4.4)**.** *Let $\mathbf{X}$ be a compact structured topological space in the same language as the finite discretely topologized structured topological space $\underset{\sim}{\mathbf{M}}$. Then $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ if and only if there is at least one morphism from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}$, and the following conditions are satisfied.*

1. *For each $x, y \in X$ with $x \neq y$, there is a morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $\alpha(x) \neq \alpha(y)$.*

2. *For each n-ary partial function symbol h and each n-tuple $(x_1, \dots, x_n) \in X^n$ outside the domain of $h^{\mathbf{X}}$, there exists a morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $(\alpha(x_1), \dots, \alpha(x_n))$ is not in the domain of $h^{\underset{\sim}{\mathbf{M}}}$.*

3. *For each n-ary relation symbol r and each $(x_1, \dots, x_n) \in X^n \backslash r^{\mathbf{X}}$, there is a morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ with $(\alpha(x_1), \dots, \alpha(x_n)) \notin r^{\underset{\sim}{\mathbf{M}}}$.*

3.2 The Stone, Priestley, and Esakia dualities

Natural duality theory's most important historical precursor is Stone-Priestley duality [53, 49, 50], the special features of which we recall presently. Recall that if $(X, \leqslant)$ is a poset, then $S \subseteq X$ is *upward closed* or an *up-set* if

$$\uparrow S := \{y \in X : \exists x \in S, x \leqslant y\}$$

coincides with S. If $(X, \leqslant, \tau)$ is an ordered topological space, we say that $(X, \leqslant, \tau)$ is a *Priestley space* provided that (X, τ) is compact and for each $x, y \in X$ with $x \nleqslant y$, there exists a clopen up-set $U \subseteq X$ such that $x \in U$ and $y \notin U$ (this demand is often called *the Priestley separation axiom*, and ordered topological spaces satisfying it are called *totally order-disconnected*). We denote by Pries the category whose objects are Priestley spaces and whose morphisms are continuous isotone functions. We also denote by $\mathsf{Dist}_{\bot\top}$ the category whose objects are bounded distributive lattices and whose morphisms are lattice homomorphisms preserving the bounds. Pries and $\mathsf{Dist}_{\bot\top}$ are dually equivalent categories via *Priestley duality*, which we describe as follows.

Recall that if $\mathbf{A} = (A, \wedge, \vee)$ is a lattice, then $\mathfrak{x} \subseteq A$ is a *filter* of $\mathbf{A}$ if is upward-closed and closed under $\wedge$. A proper, nonempty filter $\mathfrak{x}$ is called *prime* if for any $x, y \in A$, $x \vee y \in \mathfrak{x}$ implies $x \in \mathfrak{x}$ or $y \in \mathfrak{x}$. Given a bounded distributive lattice $\mathbf{A}$, we denote by $\mathcal{S}(A)$ its collection of prime filters. For a bounded distributive lattice $\mathbf{A}$ and $x \in A$, we define[6]

$$\varphi_{\mathbf{A}}(x) = \{\mathfrak{x} \in \mathcal{S}(A) : x \in \mathfrak{x}\}.$$

[6]When context makes it clear, we omit $\mathbf{A}$ and write $\varphi_{\mathbf{A}}(x)$ as $\varphi(x)$.

If τ is the topology generated by the subbase $\{\varphi_{\mathbf{A}}(x), \varphi_{\mathbf{A}}(x)^{\mathsf{c}} : x \in A\}$, then one may show that $\mathcal{S}(\mathbf{A}) := (\mathcal{S}(A), \subseteq, \tau)$ is a Priestley space.

Moving in the reverse direction, if $\mathbf{X} = (X, \leqslant, \tau)$ is a Priestley space, then we let $\mathcal{A}(X)$ be the collection of clopen up-sets of $\mathbf{X}$ and define

$$\mathcal{A}(\mathbf{X}) := (\mathcal{A}(X), \cap, \cup, \varnothing, X).$$

It is easy to see that $\mathcal{A}(\mathbf{X})$ is a bounded distributive lattice, and moreover for each bounded distributive lattice $\mathbf{A}$, the map $\varphi_{\mathbf{A}} \colon \mathbf{A} \to \mathcal{A}\mathcal{S}(\mathbf{A})$ as defined above is an isomorphism.

The maps $\mathbf{A} \mapsto \mathcal{S}(\mathbf{A})$ and $\mathbf{X} \mapsto \mathcal{A}(\mathbf{X})$ may be extended to contravariant functors by defining their action on morphisms by taking inverse images. In detail, for morphisms $h \colon \mathbf{A} \to \mathbf{B}$ in $\mathsf{Dist}_{\perp\top}$ and $\alpha \colon \mathbf{X} \to \mathbf{Y}$ in Pries, we define morphisms $\mathcal{S}(h) \colon \mathcal{S}(\mathbf{B}) \to \mathcal{S}(\mathbf{A})$ and $\mathcal{A}(\alpha) \colon \mathcal{A}(\mathbf{Y}) \to \mathcal{A}(\mathbf{X})$ by

$$\mathcal{S}(h)(\mathfrak{x}) = h^{-1}[\mathfrak{x}]$$

$$\mathcal{A}(\alpha)(U) = \alpha^{-1}[U]$$

The resulting functors $\mathcal{S}$ and $\mathcal{A}$ provide a dual equivalence of categories between $\mathsf{Dist}_{\perp\top}$ and Pries, and the unit of the corresponding adjunction is given by the sections $\varphi_{\mathbf{A}}$. The sections of the counit are given by the maps $\psi_{\mathbf{X}} \colon \mathbf{X} \to \mathcal{S}\mathcal{A}(\mathbf{X})$ defined by

$$\psi_{\mathbf{X}}(x) = \{U \in \mathcal{A}(X) : x \in U\},$$

where $\mathbf{X}$ is a Priestley space.

Note that if $\mathbf{A}$ is the bounded distributive lattice reduct of a Boolean algebra, then the prime filters of $\mathbf{A}$ coincide with its maximal proper filters (aka *ultrafilters*).

In this setting, the order on the Priestley dual $\mathcal{S}(\mathbf{A})$ is the equality relation, and $\mathcal{S}(\mathbf{A})$ may be viewed as a topological space without expanded structure. The topological spaces arising in this way are *Stone spaces*, i.e., compact Hausdorff spaces having a basis of clopen sets. Because each $\mathsf{Dist}_{\perp\top}$-morphism between Boolean algebras is a Boolean algebra homomorphism, restricting Priestley duality to Boolean algebras recovers Stone's duality between Boolean algebras and Stone spaces [53].

Priestley duality may also be restricted to obtain dualities for other important classes of bounded distributive lattices. We call a Priestley space $\mathbf{X} = (X, \leqslant, \tau)$ an *Esakia space* if for every clopen subset $U \subseteq X$, the down-set

$$\downarrow U := \{x \in X : \exists y \in U, x \leqslant y\}$$

is clopen too. A continuous isotone map $\alpha\colon \mathbf{X} \to \mathbf{Y}$ is called an *Esakia map* or *Esakia function* if for every $x \in X$ and $z \in Y$ such that $\alpha(x) \leqslant_{\mathbf{Y}} z$, there exists $y \in X$ such that $x \leqslant_{\mathbf{X}} y$ and $\alpha(y) = z$. We denote by Esa the subcategory of Pries whose objects are Esakia spaces and whose morphisms are Esakia maps.

Esakia proved in [21] that Esa and HA are dually-equivalent categories. The restrictions of the functors $\mathcal{S}$ and $\mathcal{A}$ witness this fact, with the modification that for an Esakia space $\mathbf{X}$ we define[7] for $U, V \in \mathcal{A}(X)$,

$$U \to V = (\downarrow(U \cap V^{\mathsf{c}}))^{\mathsf{c}} = (\downarrow((U^{\mathsf{c}} \cup V)^{\mathsf{c}}))^{\mathsf{c}}$$

and set $\mathcal{A}(\mathbf{X}) = (\mathcal{A}(X), \cap, \cup, \to, \varnothing, X)$.

Esakia duality is the first of many variations of Stone-Priestley duality that we will encounter in the coming pages, and in order to ease our notational burden we will use the symbols $\mathcal{S}$ and $\mathcal{A}$ for all these variations.

[7] A moment's reflection shows $U \to V = \{x \in X : U \cap \uparrow x \subseteq V\}$, which may be a more evocative presentation in connection to relational semantics.

Note that Esakia duality may itself be restricted to obtain dualities for many significant classes of Heyting algebras. Of these, we mention only its restriction to Gödel algebras: An Esakia space $(X, \leqslant, \tau)$ is the Esakia dual of a Gödel algebra if and only if $(X, \leqslant)$ is a forest[8] (see, e.g., [12]). We will employ Esakia duality for Gödel algebras in Chapter 6.

It is notable that the Stone and Priestley dualities are natural dualities in the sense of Section 3.1. To see the connection, denote by $\mathbf{2} = (\{0,1\}, \wedge, \vee, 0, 1)$ the two-element bounded distributive lattice, and by $\mathbf{2}'$ the two-element Boolean algebra (i.e., the expansion of $\mathbf{2}$ by its uniquely-determined complementation operation). Then the variety of bounded distributive lattices coincides with $\mathbb{ISP}(\mathbf{2})$, and the variety of Boolean algebras coincides with $\mathbb{ISP}(\mathbf{2}')$. Moreover, if $\mathfrak{x}$ is a prime filter of the bounded distributive lattice $\mathbf{A}$, then we may define a bounded lattice homomorphism $h_{\mathfrak{x}} \colon \mathbf{A} \to \mathbf{2}$ by

$$h_{\mathfrak{x}}(x) = \begin{cases} 1 & x \in \mathfrak{x} \\ 0 & x \notin \mathfrak{x} \end{cases}$$

and every homomorphism $\mathbf{A} \to \mathbf{2}$ is of this form for some $\mathfrak{x} \in \mathcal{S}(A)$. Moreover, given a bounded lattice homomorphism $h \colon \mathbf{A} \to \mathbf{2}$, the set $h^{-1}[1]$ is a prime filter of $\mathbf{A}$, and each prime filter of $\mathbf{A}$ is of this form. The analogous statements also hold for Boolean algebras, and in this manner one may view $\mathcal{S}$ as a hom-functor as in Section 3.1. Likewise, $\mathcal{A}$ may be presented in terms of the two-element linearly-ordered Priestley space (or two-element Stone space). Note that both the Stone and Priestley dualities may be obtained by using the NU strong duality theorem. In contrast, Esakia duality is *not* a natural duality because HA is not $\mathbb{ISP}(\mathbf{A})$ for

[8] A poset P is a *forest* if $\uparrow x$ is totally-ordered for any $x \in P$.

any finite algebra $\mathbf{A}$ (or even any finite collection of finite algebras). This difficulty persists even if one restricts one's attention to Gödel algebras (but see [17, 12]).

From the perspective of natural duality theory, it is easy to see that Priestley duality may be modified in order to account for the omission of one or both bounds from the algebraic signature. Let $\mathsf{Dist}_\top$ be the category of distributive lattices with a designated greatest element (and possibly missing a least element). A *pointed Priestley space* is a structure of the form $\mathbf{X} = (X, \leqslant, \top, \tau)$, where $(X, \leqslant, \tau)$ is a Priestley space and $\top$ is a constant designating the greatest element of $(X, \leqslant)$. We denote the category of pointed Priestley spaces (with continuous isotone maps preserving $\top$) by pPries. The categories $\mathsf{Dist}_\top$ and pPries are dually equivalent via the functors $\mathcal{S}$ and $\mathcal{A}$, subject to the following modifications:

1. For an object $\mathbf{A}$ of $\mathsf{Dist}_\top$, we let $\mathcal{S}(A) = \{\mathfrak{x} : \mathfrak{x} \text{ is a prime filter of } \mathbf{A} \text{ or } \mathfrak{x} = A\}$.

2. For an object $\mathbf{X}$ of pPries, we let $\mathcal{A}(X) = \{U \subseteq X : U \text{ is a clopen and } U \neq \varnothing\}$.

Similar comments apply to the omission of the bottom bound or both bounds from the signature. Each of these modifications of Priestley duality may be found by application of the NU duality theorem. We sometimes refer to the elements of $\{\mathfrak{x} : \mathfrak{x} \text{ is a prime filter of } \mathbf{A} \text{ or } \mathfrak{x} = A\}$ as *generalized prime filters* of $\mathbf{A}$.

Priestley duality for top-bounded distributive lattices may be restricted to give a duality for Brouwerian algebras, just as Priestley duality in its fully-bounded incarnation may be restricted to give Esakia duality for Heyting algebras. The category BrA is hence dually equivalent to the category pEsa of *pointed Esakia spaces* with *pointed Esakia maps*. The pointed Esakia spaces corresponding to relative Stone algebras are precisely the pointed Esakia spaces whose order reducts are top-bounded forests (aka trees).

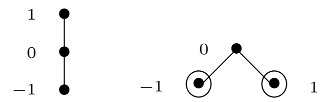


Figure 3.1: Hasse diagrams for the different personalities of the object $\mathbf{K}$

3.3 The Davey-Werner duality

The variety of Kleene algebras (see Section 2.2) coincides with $\mathbb{ISP}(\mathbf{K})$, where

$$\mathbf{K} = (\{-1, 0, 1\}, \wedge, \vee, \neg, -1, 1)$$

is the expansion of the normal distributive i-lattice $\mathbf{D}_3$ by constants designating the least and greatest elements. Davey and Werner gave a strong natural duality for Kleene algebras in [19], using $\mathbf{K}$ as a dualizing object. Under the Davey-Werner duality, the alter ego of $\mathbf{K}$ is

$$\underset{\sim}{\mathbf{K}} = (\{-1, 0, 1\}, \leqslant, Q, K_0, \tau),$$

where $\leqslant$ is the partial order determined by $-1 < 0$ and $1 < 0$, Q is the binary relation given by xQy iff $(x, y) \notin \{(-1, 1), (1, -1)\}$, $K_0 = \{-1, 1\}$, and τ is the discrete topology on $\{-1, 0, 1\}$ (see Figure 3.1). The following provides a useful external description of $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{K}})$ (see [14, p. 107] and [19]).

Proposition 3.3.1. *$(X, \leqslant, Q, X_0, \tau)$ is an isomorphic copy of a closed substructure of a nonempty power of $\underset{\sim}{\mathbf{K}}$ if and only if all the following hold.*

1. *$(X, \leqslant, \tau)$ is a Priestley space,*

2. *Q is a binary relation that is closed in X^2,*

3. *X_0 is a closed subspace, and*

4. *For all $x, y, z \in X$,*

   (a) xQx,

   (b) xQy *and* $x \in X_0 \implies x \leqslant y$,

   (c) xQy *and* $y \leqslant z \implies zQx$.

We say that $(X, \leqslant, Q, X_0, \tau)$ is a *Kleene space* if it satisfies the conditions given in Proposition 3.3.1, and denote the category of Kleene spaces with continuous structure-preserving morphisms by KS. From the above, KA and KS are dually equivalent categories.

Later on, we will restrict the Davey-Werner duality to a subcategory of KS that provides a duality for bounded Sugihara monoids. To get a duality for Sugihara monoids *tout court*, we need a variant of the Davey-Werner duality for normal distributive i-lattices (i.e., we must drop bounds from the signature). This variant of the Davey-Werner duality originally comes from the author's [24]. Recall that $\mathsf{NDIL} = \mathbb{ISP}(\mathbf{D}_3)$, where $\mathbf{D}_3 = (\{-1, 0, 1\}, \wedge, \vee, \neg)$ is the three-element i-lattice chain with one zero.

Theorem 3.3.2. *Let $\underset{\sim}{\mathbf{D}}_3 = (\{-1, 0, 1\}, \leqslant, Q, D_0, 0, \tau)$, where $\leqslant$ is the partial order determined by $-1 < 0$ and $1 < 0$, D_0 is the unary relation $\{-1, 1\}$, Q is the binary relation given by xQy iff $(x, y) \notin \{(-1, 1), (1, -1)\}$, and 0 is a constant designating the greatest element with respect to $\leqslant$. Then $\underset{\sim}{\mathbf{D}}_3$ dualizes $\mathbf{D}_3$, and this duality is strong.*

Proof. We will use Corollary 3.1.2. Let $D_3 = \{-1, 0, 1\}$ be the universe of $\mathbf{D}_3$. Direct computation verifies that the following are the carriers of subalgebras $\mathbf{D}_3^2$:

$$\{0\}, \Delta_{D_0}, \leqslant \cap (D_0 \times D_3), \geqslant \cap (D_3 \times D_0), D_0 \times D_3, D_3 \times D_0, D_3^2, \Delta_{D_3}, \leqslant,$$

$$\geqslant, Q, D_0 \times \{0\}, \{0\} \times D_0, D_3 \times \{0\}, \{0\} \times D_3, D_0^2,$$

where Δ_S denotes the equality relation on a given set S. It is easy to see that $\{\leqslant, D_0, Q, 0\}$ entails the above collection of relations (see, e.g., [14, Section 2.4]).

We next compute $\mathcal{P}_1$:

$$h_0 \colon \{0\} \to \mathbf{D}_3 \text{ defined by } h_0(0) = 0$$

$$h_1 \colon \{-1, 1\} \to \mathbf{D}_3 \text{ defined by } h_1(-1) = h_1(1) = 0$$

$$h_2 \colon \{-1, 1\} \to \mathbf{D}_3 \text{ defined by } h_2(-1) = -1 \text{ and } h_2(1) = 1$$

$$h_3 \colon \mathbf{D}_3 \to \mathbf{D}_3 \text{ defined by } h_3(-1) = h_3(0) = h_3(1) = 0$$

$$h_4 \colon \mathbf{D}_3 \to \mathbf{D}_3 \text{ defined by } h_4(x) = x \text{ for all } x \in \{-1, 0, 1\}$$

The graphs of the above are given by

$$\mathrm{grph}(h_0) = \{(0,0)\} = \{0\} \times \{0\}$$

$$\mathrm{grph}(h_1) = \{(-1,0), (1,0)\} = D_0 \times \{0\}$$

$$\mathrm{grph}(h_2) = \{(-1,1), (1,1)\} = \Delta_{D_0}$$

$$\mathrm{grph}(h_3) = \{(-1,0), (0,0), (1,0)\} = D_3 \times \{0\}$$

$$\mathrm{grph}(h_4) = \{(-1,-1), (0,0), (1,1)\} = D_3 \times \{0\}$$

This proves that $\{\leqslant, 0, D_0, Q\}$ entails $\{0, \mathcal{P}_1, \mathcal{B}_2\}$.

To conclude the proof, it suffices to show that $\{0\}$ hom-entails $\{h_0, h_1, h_2, h_3, h_4\}$. Theorem 3.1.3(3)(b) guarantees that we may delete h_0, h_1, and h_2 since h_3 and h_4 extend them. Since h_4 is the identity endomorphism, it is hom-entailed by any set of partial operations. Because h_3 is the constant endomorphism associated with 0, it is hom-entailed by the constant 0. This proves the result. $\Box$

We will provide an external characterization of the structured topological spaces in $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}}_3)$. This characterization and the arguments supporting it amount to those in [14, Theorem 4.3.10], but for completeness—and because they will be useful later—we recite them here. The structured topological spaces of interest are the following.

Definition 3.3.3. *A structure $(X, \leqslant, Q, D, \top, \tau)$ is a* pointed Kleene space *if:*

1. *$(X, \leqslant, \tau)$ is a Priestley space whose greatest element is $\top \notin D$,*
2. *Q is a binary relation that is closed in X^2,*
3. *D is a closed subspace, and*
4. *For all $x, y, z \in X$,*
   - *(a) xQx,*
   - *(b) xQy and $x \in D \implies x \leqslant y$,*
   - *(c) xQy and $y \leqslant z \implies zQx$.*

Lemma 3.3.4. *Let $\mathbf{X} = (X, \leqslant, Q, D, \top, \tau)$ be a pointed Kleene space. Then $\mathbf{X}$ satisfies the following.*

1. *Q is symmetric.*
2. *If $x \leqslant y$, then yQx.*
3. *If $y \leqslant x$ and $x \in D$, then $y = x$.*
4. *If $x \leqslant y$ and $x \leqslant z$, then yQz.*

5. *If $x \in D$, then xQy if and only if $x \leqslant y$.*

Proof. Each of the above properties hold in every Kleene space by [14, p. 107], and therefore hold in every pointed Kleene space as well. $\square$

Let X be a set. For each $U, V \subseteq X$ with $U \cup V = X$, we define a function $C_{U,V} \colon X \to \{-1, 0, 1\}$ by

$$C_{U,V}(x) = \begin{cases} 1, & \text{if } x \notin V \\ 0, & \text{if } x \in U \cap V \\ -1, & \text{if } x \notin U \end{cases}$$

Note that the map $C_{U,V}$ is well-defined because $U \cup V = X$.

Lemma 3.3.5. *Let $\mathbf{X} = (X, \leqslant, Q, D, \top, \tau)$ be a structure in the language of pointed Kleene spaces, and let $U, V \subseteq X$ with $U \cup V = X$. Then $C_{U,V}$ is a continuous structure-preserving morphism from $\mathbf{X}$ to $\underset{\sim}{\mathbf{D}}_3$ if and only if U, V are clopen up-sets with $(X \backslash U \times X \backslash V) \cap Q = \varnothing$ and $U \cap V \subseteq D^{\mathsf{c}}$.*

Proof. Suppose that $C_{U,V} \colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}_3$ is a morphism. Then U and V are clopen up-sets because they are the inverse images of clopen up-sets, viz. $U = C_{U,V}^{-1}(\{0, 1\})$ and $V = C_{U,V}^{-1}(\{-1, 0\})$. Observe that if $x, y \in X$ with $x \notin U$ and $y \notin V$, then $C_{U,V}(x) = -1$ and $C_{U,V}(y) = 1$ are not Q-related in $\underset{\sim}{\mathbf{D}}_3$. If follows that xQy fails in $\mathbf{X}$, whence $(X \backslash U \times X \backslash V) \cap Q = \varnothing$. To see that $U \cap V \subseteq D^{\mathsf{c}}$, notice that if $x \in U \cap V$ then $C_{U,V}(x) = 0 \notin D_0$. This gives $x \notin D$ since $C_{U,V}$ is structure-preserving. Hence $x \in D^{\mathsf{c}}$, and $U \cap V \subseteq D^{\mathsf{c}}$ follows.

To prove the converse, assume that U and V are clopen up-sets with

$$(X \backslash U \times X \backslash V) \cap Q = \varnothing \text{ and } U \cap V \subseteq D^{\mathsf{c}}.$$

We will prove that $C_{U,V}$ is a continuous structure-preserving morphism. For continuity, it suffices to notice that

$$
\begin{aligned}
C_{U,V}^{-1}[\{0\}] &= U \cap V \\
C_{U,V}^{-1}[\{-1\}] &= U^{\mathsf{c}} \\
C_{U,V}^{-1}[\{1\}] &= V^{\mathsf{c}}
\end{aligned}
$$

are all open in $\mathbf{X}$.

For the preservation of the order relation, let $x, y \in X$ with $x \leqslant y$. Were $C_{U,V}(y) = 0$, we would have $C_{U,V}(x) \leqslant C_{U,V}(y)$ because 0 is the greatest element of $\underset{\sim}{\mathbf{D}}_3$. Were $C_{U,V}(y) = 1$, then by definition $y \notin V$. Because V is an up-set, this implies $x \notin V$ as well, and hence $C_{U,V}(x) = 1$. An identical argument shows that if $C_{U,V}(y) = -1$, then $C_{U,V}(x) = -1$. Thus $C_{U,V}$ preserves $\leqslant$.

For the preservation of Q, let $x, y \in X$ with $C_{U,V}(y) = 1$ and $C_{U,V}(x) = -1$. Then $y \notin V$ and $x \notin U$, so we have $(x, y) \in X \backslash U \times X \backslash V$. It follows that $(x, y) \notin Q$ because $(X \backslash U \times X \backslash V) \cap Q = \varnothing$, whence by taking the contrapositive we have that xQy implies $C_{U,V}(x)\ Q\ C_{U,V}(y)$.

For the preservation of D, let $x \in D$. Then $x \notin U \cap V$ since $U \cap V \subseteq D^{\mathsf{c}}$, and thus $C_{U,V}(x) = -1$ or $C_{U,V}(x) = 1$, i.e., $C_{U,V}(x) \in D_0$.

Lastly, for the preservation of $\top$, note that U, V being up-sets gives $\top \in U \cap V$. Then $C_{U,V}(\top) = 0$ by the definition of $C_{U,V}$, and 0 is the greatest element of $\underset{\sim}{\mathbf{D}}_3$. This settles the proof. □

Lemma 3.3.6. *Let $\mathbf{X} = (X, \leqslant, Q, D, \top, \tau)$ be a pointed Kleene space and let $\alpha \colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}_3$ be a continuous structure-preserving morphism. Then there exist clopen up-sets $U, V \subseteq X$ such that $\alpha = C_{U,V}$.*

Proof. Set $U := \alpha^{-1}[\{0,1\}]$ and $V := \alpha^{-1}[\{-1,0\}]$. Then U and V are clopen up-sets, being the inverse images of clopen up-sets. Moreover, $C_{U,V}(x) = \alpha(x)$ for all $x \in X$. $\square$

Theorem 3.3.7. $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}}_3)$ *is exactly the class of pointed Kleene spaces.*

Proof. We apply the preservation and separation theorems. Note that $\underset{\sim}{\mathbf{D}}_3$ is a pointed Kleene space, whence Theorem 3.1.4 gives that $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}}_3)$ consists of pointed Kleene spaces.

For the reverse inclusion, we apply Theorem 3.1.5. Let $\mathbf{X} = (X, \leqslant, Q, D, \top, \tau)$ be a pointed Kleene space. Firstly, let $x, y \in X$ so that xQy fails. Note that $\{z : zQx\}$ is an up-set by Definition 3.3.3(4)(c). Moreover, since Q is closed in X^2 and (X, τ) is compact, we have also that $\{z : zQx\}$ is closed (i.e., since the projection maps are closed maps in this setting). As $\mathbf{X}$ is a Priestley space, $\{z : zQx\}$ is hence the intersection of clopen up-sets. Because $y \notin \{z : zQx\}$, there exists a clopen up-set U with $y \notin U$ and $z \in U$ for every zQx. Set

$$W := \{w \in X : (\forall z \in U^{\mathsf{c}})(zQw \text{ fails})\}.$$

Then W is open as a consequence of Q being closed and U^{c} being compact, and is down-set by Definition 3.3.3(4)(c). By Lemma 3.3.4(5), we have also that $D \cap U^{\mathsf{c}} \subseteq W$. There is hence a clopen down-set $W' \subseteq W$ such that $\{x\} \cup (D \cap U^{\mathsf{c}}) \subseteq W'$. Setting $V = (W')^{\mathsf{c}}$, we have that V is a clopen up-set with $x \notin V$ and $U \cap V \subseteq D^{\mathsf{c}}$. Moreover, $(X \backslash U \times X \backslash V) \cap Q = \varnothing$. It follows that $C_{U,V}$ separates x and y.

Secondly, let $x, y \in X$ with $x \nleqslant y$. In the case that xQy fails, we may use the separating morphism constructed above. In the case that xQy, we have that $x \notin D$ by Lemma 3.3.4(5), and from 3.3.4(3) it follows that $x \nleqslant z$ for each $z \in D$. There hence exists a clopen up-set U disjoint from $\{y\} \cup D$ with $x \in U$. The morphism $C_{U,X}$ then separates x and y.

Thirdly, let $x, y \in X$ with $x \neq y$. Then $x \nleqslant y$ or $y \nleqslant x$, so x and y may be separated by the above.

Fourthly, if $x \notin D$, then we consider two cases. First, if $D = \varnothing$, then use $C_{X,X}$. If $D \neq \varnothing$, then for any $y \in D$ we have that $x \nleqslant y$. In this case, we may use the separating morphism constructed above. $\square$

The following is immediate by combining the results above.

Corollary 3.3.8. NDIL *is dually equivalent to the category* pKS *is pointed Kleene spaces and continuous structure-preserving morphisms.*

In the remainder of our work, we will reserve the symbols $\mathcal{D}$ and $\mathcal{E}$ for the functors of the Davey-Werner duality (whether for KA or NDIL).

We conclude this section by recalling some well-known technical results that prove useful for working with the topologies of (pointed) Kleene spaces.

Lemma 3.3.9 ([14, Lemma B.6, p. 340])**.** *Let A be an index set and $\mathbf{L} \in \{\mathbf{D}_3, \mathbf{K}\}$. Consider $\mathbf{L}^A$ as a topological space endowed with the product topology. For each $a \in A$ and $l \in \{-1, 0, 1\}$, let $U_{a,l} = \{x \in \mathbf{L}^A : x(a) = l\}$. Then*

$$\{U_{a,l} : a \in A \text{ and } l \in \{-1, 0, 1\}\}$$

is a clopen subbasis for the topology on $\mathbf{L}^A$.

Given an $\mathbf{A} \in \mathsf{NDIL} \cup \mathsf{KA}$, the Davey-Werner dual of $\mathbf{A}$ has topology induced as a subspace of $\mathbf{L}^A$ as above. Hence from the previous lemma we obtain

Lemma 3.3.10. *Let $\mathbf{A} \in \mathsf{NDIL} \cup \mathsf{KA}$. Then the sets $U_{a,l} = \{h \in \mathcal{D}(\mathbf{A}) : h(a) = l\}$, where $l \in \{-1, 0, 1\}$ and $a \in A$, give a clopen subbasis for the topology on $\mathcal{D}(\mathbf{A})$.*

3.4 Extended Priestley duality for residuated structures

Of the topological dualities we have seen so far, only Esakia duality provides a dual equivalence between a category of structured topological spaces and a category of residuated algebras. Esakia duality is a restriction of Priestley duality, and this method of obtaining a dual equivalence relies on the fact that Heyting algebras are uniquely determined by their lattice reducts. For most classes of distributive residuated lattice-based structures, this method is hopeless: A single lattice typically admits many different residuated expansions.[9] We show in Chapter 6 that Sugihara monoids and bounded Sugihara monoids are uniquely determined by their reduct in NDIL, and enjoy an Esakia-like duality by restricting the Davey-Werner duality. Except for the special cases of Heyting algebras, Sugihara monoids, and some of their expansions and reducts, we must use another method to get topological dualities for (distributive) residuated algebras—namely, augmenting the structure of Priestley duals. We turn to this *extended Priestley duality* in the present section.

The ideas discussed here descend from Jónsson and Tarski's celebrated work on Boolean algebras with operators [39, 40] and Hansoul's duality theory for them [36]. In the style depicted here, they come from various studies of Urquhart. This body of work is probably most thoroughly synthesized in in Urquhart's [56]. We draw most of our exposition from the latter, but also rely on Galatos's exposition [28].[10] We also refer to [13, 11, 35] for further information.

[9]To get a sense of the scale of this problem even for finite algebras, there are just two lattices (both distributive) based on a four element set. Computer-assisted enumeration shows that up to isomorphism there are 20 residuated lattices on four elements. Up to isomorphism, there are 1116 residuated binars on four elements.

[10]We note that each [56] and [28] is more general than the other in different directions. Urquhart accounts for nonassociative residuated structures, but includes only one of the two residuals in the language and adds an additional unary operation $\neg$ satisfying the De Morgan laws. Galatos includes both residuals in his treatment, but assumes associativity and does not include a negation-like operation. Because our interest is commutative residuated lattices (where the two residuals coincide) with negation, Urquhart's treatment is most suitable for us.

We first provide some notational conventions. Suppose that $(\mathbf{X}, R)$ is a structure consisting of a Priestley space $\mathbf{X}$ and a ternary relation $R \subseteq X^3$. Given $U, V \subseteq X$, define

$$U \cdot V = \{z \in X : (\exists x \in U, y \in V) R(x, y, z)\}$$

$$U \to V = \{x \in X : (\forall y, z \in X)(R(x, y, z) \text{ and } y \in U) \implies z \in V)\}$$

The following specializes the dual spaces defined in [56] to the commutative and associative case. Recall that if $\mathbf{X}$ is a Priestley space, then $\mathcal{A}(X)$ denotes the set of clopen subsets of $\mathbf{X}$ (see Section 3.2).

Definition 3.4.1. *A structure* $(\mathbf{X}, R, {}^*, I)$ *is a* residuated Priestley space *if* $\mathbf{X}$ *is a Priestley space,* $R \subseteq X^3$, * *is a unary operation on* X, *and* $I \subseteq X$, *and for all* $x, y, z, w, x', y', z' \in X$*:*

1. *There exists* $u \in X$ *such that* $R(x, y, u)$ *and* $R(u, z, w)$ *if and only if there exists* $v \in X$ *such that* $R(y, z, v)$ *and* $R(x, v, w)$.
2. $R(x, y, z)$ *if and only if* $R(y, x, z)$.
3. *If* $x' \leqslant x$, $y' \leqslant y$, *and* $z \leqslant z'$ *and* $R(x, y, z)$, *then* $R(x', y', z')$.
4. *If* $R(x, y, z)$ *fails, then for some* $U, V \in \mathcal{A}(X)$ *we have* $x \in U$, $y \in V$, *and* $z \notin U \cdot V$.
5. *For all* $U, V \in \mathcal{A}(X)$, *the sets* $U \cdot V$ *and* $U \to V$ *are clopen.*
6. $I \in \mathcal{A}(X)$, *and* $U \cdot I = I \cdot U = U$ *for all* $U \in \mathcal{A}(X)$.
7. * *is continuous and antitone.*

If $\mathbf{X}_1 = (X_1, \leqslant_1, R_1, I_1, {}^*, \tau_1)$ *and* $\mathbf{X}_2 = (X_2, \leqslant_2, R_2, I_2, {}^*, \tau_2)$ *are residuated Priestley spaces, a map* $\alpha \colon X_1 \to X_2$ *is a* bounded morphism *if it satisfies the following five conditions.*

1. *α is continuous and isotone.*

2. *$R_1(x, y, z)$ implies $R_2(\alpha(x), \alpha(y), \alpha(z))$.*

3. *If $R_2(u, v, \alpha(z))$, then there are $x, y \in X_1$ such that $u \leqslant \alpha(x)$, $v \leqslant \alpha(y)$, and $R_1(x, y, z)$.*

4. *If $R_2(\alpha(x), v, w)$, then there are $v, w \in X_1$ such that $y \leqslant \alpha(v)$, $\alpha(w) \leqslant z$, and $R_1(x, v, w)$.*

5. *$\alpha^{-1}[I_2] = I_1$.*

6. *$\alpha(x^*) = \alpha(x)^*$.*

Residuated Priestley spaces and bounded morphisms form a category, which we denote by $\mathsf{RL}^{\tau}_{\bot}$.

Theorem 3.4.2 ([56]). *The category of bounded distributive commutative residuated lattices $\mathsf{RL}_{\bot}$ with De Morgan negation is dually equivalent to $\mathsf{RL}^{\tau}_{\bot}$.*

To describe how to augment the functors $\mathcal{A}$ and $\mathcal{S}$ so as to obtain the duality of Theorem 3.4.2, we introduce some more notation. For $\mathbf{A}$ a residuated lattice, the *complex product* of filters $\mathfrak{x}$, $\mathfrak{y}$ of $\mathbf{A}$ is the set $\mathfrak{x} \cdot \mathfrak{y} = \{xy : x \in \mathfrak{x}, y \in \mathfrak{y}\}$ and the *filter product* of $\mathfrak{x}$ and $\mathfrak{y}$ is

$$\mathfrak{x} \bullet \mathfrak{y} = \uparrow(\mathfrak{x} \cdot \mathfrak{y}) = \{z \in A : (\exists x \in \mathfrak{x}, y \in \mathfrak{y})(xy \leqslant z)\}.$$

Obviously, if $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ are filters, then $\mathfrak{x} \cdot \mathfrak{y} \subseteq \mathfrak{z}$ if and only if $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$.

Let $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, e, \bot, \top, \neg)$ be a bounded distributive CRL expanded by a negation $\neg$ that satisfies the De Morgan laws. Moreover, let $\mathbf{L}$ be its reduct in $\mathsf{Dist}_{\bot\top}$. Define a ternary relation R on $\mathcal{S}(L)$ by

$$R(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \text{ iff } \mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}.$$

Moreover, set

$$I := \{\mathfrak{x} \in \mathcal{S}(L) : e \in \mathfrak{x}\},$$

$$\mathcal{S}(\mathbf{A}) := (\mathcal{S}(\mathbf{L}), R, I),$$

$$\mathfrak{x}^* := \{x \in A : \neg x \notin \mathfrak{x}\}$$

The operation * defined above is sometimes called the *Routley star* (see [52, 51]).

For the other functor, if $\mathbf{X} = (X, \leqslant, R, I, ^*, \tau)$ is a residuated Priestley space, set

$$\mathcal{A}(\mathbf{X}) := (\mathcal{A}(X, \leqslant, \tau), \cdot, \to, I, \neg),$$

where the operations $\cdot, \to$ are defined for $U, V \in \mathcal{A}(X, \leqslant, \tau)$ as above, and

$$\neg U = \{x \in X : x^* \notin U\}.$$

The foregoing augmentations of $\mathcal{S}$ and $\mathcal{A}$ give the dual equivalence between the category of bounded distributive CRLs expanded by a negation and $\mathsf{RL}_\bot^\tau$.

Remark 3.4.3. Observe that if $\mathbf{A}$ is a bounded CRL and the negation operation $\neg$ treated above is defined by $x \mapsto x \to \bot$ (e.g., as in MTL-algebras), then the inclusion of * on the dual space is extraneous. In this situation, we will often drop * from the signature.

[56] and [11] give correspondences between many equational properties of residuated structures and their dual spaces, allowing us to formulate extrinsic axiomatizations of the dual spaces corresponding to the residuated lattices of interest. In particular, one may explicitly axiomatize the categories $\mathsf{SM}_\bot^\tau$ and MTL^τ that provide extended Priestley duals of algebras in $\mathsf{SM}_\bot$ and MTL. However, we will not need to employ an explicit description of these categories, and are content that $\mathsf{SM}_\bot^\tau$ and

MTL^τ exist and are dually equivalent to $\mathsf{SM}_\perp$ and MTL, respectively, via restrictions of the functors $\mathcal{S}$ and $\mathcal{A}$.

3.4.1 Dropping lattice bounds

So far, we have followed previous authors by formulating extended Priestley duality in terms of bounded residuated structures. However, we need a variant of extended Priestley duality for GMTL for our work in Chapter 8, and we construct the aforementioned variant in this section. The results of this section come from the author's work in [27], and were inspired by [37].

Denote by $\mathsf{MTL}_{\mathsf{div}}$ the full subcategory of MTL whose objects have no zero divisors.

Theorem 3.4.4. *$\mathsf{MTL}_{\mathsf{div}}$ and GMTL are equivalent.*

Proof. We define a functor $(-)_0 \colon \mathsf{GMTL} \to \mathsf{MTL}_{\mathsf{div}}$ as follows. Given an object $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 1)$ of GMTL, we define an algebra $\mathbf{A}_0$ on the carrier $A \cup \{0\}$, where $0 \notin A$ is a new element.[11] The lattice order on $\mathbf{A}_0$ is uniquely determined by setting $0 < a$ for all $a \in A$. For the multiplication and its residual, we define a new operation $*$ on $A \cup \{0\}$ by

$$a * b = \begin{cases} a \cdot b & a, b \in A \\ 0 & a = 0 \text{ or } b = 0 \end{cases}$$

[11] Note that $\mathbf{A}_0$ is the ordinal sum $\mathbf{2} \oplus \mathbf{A}$ of $\mathbf{2}$ and $\mathbf{A}$.

This uniquely determines a residual $\twoheadrightarrow$ of $*$ given by

$$a \twoheadrightarrow b = \begin{cases} a \to b & a, b \in A \\ 0 & b = 0 \text{ and } a \in A \\ 1 & a = 0 \end{cases}$$

One can readily check that $\mathbf{A}_0 = (A \cup \{0\}, \wedge, \vee, *, \twoheadrightarrow, 1, 0)$ is a bounded, distributive, integral CRL. Moreover, $\mathbf{A}_0 \models (x \to y) \vee (y \to x) = 1$, whence $\mathbf{A}_0$ is an MTL-algebra. Since $\mathbf{A}$ is a subalgebra of $\mathbf{A}_0$, we have $\mathbf{A}$ is an object of $\mathsf{MTL_{div}}$.

Given GMTL-algebras $\mathbf{A}$ and $\mathbf{B}$ and a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$, define a map $h_0\colon \mathbf{A}_0 \to \mathbf{B}_0$ by

$$h_0(x) = \begin{cases} h(x) & x \in A \\ 0^{\mathbf{B}} & x = 0^{\mathbf{A}} \end{cases}$$

Then h is a homomorphism of MTL-algebras. It is easy to check that $(-)_0$ is functorial.

Note that if $\mathbf{A}$ and $\mathbf{B}$ are objects of GMTL and $h\colon \mathbf{A}_0 \to \mathbf{B}_0$ is a homomorphism, then the restriction $h{\restriction_A}$ of h to A is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$, and $(h{\restriction_A})_0 = h$. This shows that $(-)_0$ is full. It is obviously faithful as well.

To see that $(-)_0$ is essentially surjective, let $\mathbf{A}$ be an object of $\mathsf{MTL_{div}}$. Observe that $A \backslash \{0\}$ is closed under $\cdot$ by the fact that $\mathbf{A}$ has no zero divisors. Moreover, since $y \leqslant x \to y$ for any $x, y \in A$, we have $y \leqslant x \to y \neq 0$ whenever $y \neq 0$. This shows that $A \backslash \{0\}$ is closed under $\to$. Since $x \cdot y \leqslant x \wedge y$ for any $x, y \in A$, $A \backslash \{0\}$ is closed under the lattice connectives too. It follows that $A \backslash \{0\}$ is the carrier of a $(\wedge, \vee, \cdot, \to, 1)$-subalgebra $\mathbf{A}'$ of $\mathbf{A}$. $\mathbf{A}'$ is a GMTL-algebra, and $\mathbf{A}'_0 \cong \mathbf{A}$.

The above proves that $(-)_0$ is full, faithful, and essentially surjective, and therefore witnesses an equivalence of categories. $\square$

The dual equivalence between MTL and MTL^τ may be restricted to obtain a dual equivalence between $\mathsf{MTL}_{\mathsf{div}}$ and the corresponding full subcategory $\mathsf{MTL}^\tau_{\mathsf{div}}$ of MTL^τ. Following Theorem 3.4.4, GMTL is dually equivalent to $\mathsf{MTL}^\tau_{\mathsf{div}}$ by composing the relevant functors. Spelling this out, let $\mathbf{A}$ be a GMTL-algebra. By the above, $\mathbf{A}_0$ is an MTL-algebra with dual $\mathcal{S}(\mathbf{A}_0)$ in $\mathsf{MTL}^\tau_{\mathsf{div}}$. Notice that by construction A is a prime filter of $\mathbf{A}_0$, giving that the dual space $\mathcal{S}(\mathbf{A}_0)$ has a greatest element. If $\mathbf{A}$ and $\mathbf{B}$ are GMTL-algebras and $h\colon \mathbf{A} \to \mathbf{B}$ is a homomorphism, the $\mathcal{S}(h_0)$ preserves the greatest element of $\mathcal{S}(\mathbf{B}_0)$ because $h_0^{-1}[B] = A$.

For a top-bounded object $\mathbf{X} = (X, \leqslant, \tau, R, E)$ of MTL^τ, the set of nonempty members of $\mathcal{A}(\mathbf{X})$ is closed under the operations $\cap$, $\cup$, $\cdot$, and $\to$, and also $E \neq \varnothing$. Consequently, the nonempty clopen up-sets of $\mathbf{X}$ are the universe of a $(\wedge, \vee, \cdot, \to, 1)$-subalgebra of $\mathcal{A}(\mathbf{X})$. This subalgebra is a GMTL-algebra. Also, if $\top_X$ and $\top_Y$ are the greatest elements of top-bounded MTL^τ-objects $\mathbf{X}$ and $\mathbf{Y}$ and $\alpha\colon \mathbf{X} \to \mathbf{Y}$ is a morphism preserving the greatest element, then for each $U \in \mathcal{A}(Y)$ we have $\top_Y \in \mathcal{A}(\alpha)(U)$ gives that $\mathcal{A}(\alpha)(U) \neq \varnothing$. This demonstrates that such a map α restricts to the to GMTL-algebras of nonempty clopen up-sets of $\mathbf{X}$ and $\mathbf{Y}$.

Definition 3.4.5. *Let* GMTL^τ *be the category with*

- *objects given by structures* $(\mathbf{X}, R, I, \top)$*, where* $(\mathbf{X}, R, I)$ *is an object of* MTL^τ *with maximum element* $\top$*.*
- *morphisms given by maps* $\alpha\colon (\mathbf{X}_1, R_1, I_1, \top_1) \to (\mathbf{X}_2, R_2, I_2, \top_2)$ *between objects of* GMTL^τ*, where* α *is a bounded morphism* $(\mathbf{X}_1, R_1, I_1) \to (\mathbf{X}_2, R_2, I_2)$ *and* $\alpha(\top_1) = \top_2$*.*

As an immediate consequence of the work in this section, we have:

Theorem 3.4.6. GMTL *and* GMTL^τ *are dually equivalent.*

In analogy to Priestley duality for $\mathsf{Dist}_{\top}$, we once again use the symbols $\mathcal{A}$ and $\mathcal{S}$ for the duality between GMTL and GMTL^{τ}. In particular, for an object $\mathbf{A}$ and a morphism h of GMTL, by $\mathcal{S}(\mathbf{A})$ and $\mathcal{S}(h)$ we respectively mean $\mathcal{S}(\mathbf{A}_0)$ and $\mathcal{S}(h_0)$ (where the latter two occurrences of $\mathcal{S}$ refer to the variant of this functor for MTL).

Chapter 4

Functional dualities for residuated structures

Although Esakia duality is a standard tool in the study of Heyting algebras, extended Priestley duality in the style of Section 3.4 has attracted comparatively few applications to more general kinds of distributive residuated lattices. This is probably a consequence of the complexity of residuated Priestley spaces vis-à-vis Esakia spaces, and in particular the conceptual hurdle of working with the ternary relation dual to the residuated operations. Sometimes this difficulty may be ameliorated because the ternary relations of a class of residuated Priestley spaces has a particularly simple form. This chapter explores one such situation, focusing on residuated Priestley spaces where the relation dualizing multiplication can be understood as a (sometimes partially-defined) function. Section 4.1 explores this phenomenon in the context of semilinear residuated lattices, and descends from the author's work in [27]. Section

4.2 adopts a more abstract approach to the functionality phenomenon, and comes from the author's [26].

4.1 Functional duality for semilinear residuated lattices

Recall that for any residuated lattice $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, e)$ and $\mathfrak{x}$ and $\mathfrak{y}$ filters of $\mathbf{A}$, the filter product of $\mathfrak{x}$ and $\mathfrak{y}$ is

$$\mathfrak{x} \bullet \mathfrak{y} = \{z \in A : (\exists x \in \mathfrak{x}, y \in \mathfrak{y})\ xy \leqslant z\}.$$

The next lemma provides an essential result for working with $\bullet$.

Lemma 4.1.1. *[28, Lemmas 6.8 and 6.9] Let $\mathbf{A}$ be a residuated lattice and let $\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$ be filters of $\mathbf{A}$. Then we have:*

1. *$\mathfrak{x} \bullet \mathfrak{y}$ is a filter of $\mathbf{A}$.*
2. *If $\mathbf{A}$ has a distributive lattice reduct, $\mathfrak{z}$ is prime, and $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$, then there exist prime filters $\mathfrak{x}'$ and $\mathfrak{y}'$ of $\mathbf{A}$ such that $\mathfrak{x} \subseteq \mathfrak{x}'$, $\mathfrak{y} \subseteq \mathfrak{y}'$, $\mathfrak{x}' \bullet \mathfrak{y} \subseteq \mathfrak{z}$, and $\mathfrak{x} \bullet \mathfrak{y}' \subseteq \mathfrak{z}$.*

The operation $\bullet$ on the filter lattice of $\mathbf{A}$ restricts to $\mathcal{S}(A) \cup \{A\}$ in some contexts. Recall the distributive laws $(\backslash\vee)$ and $(\vee/)$ from Section 2.1.1.

Lemma 4.1.2. *Let $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, e)$ be a residuated lattice and let $\mathfrak{x}, \mathfrak{y}$ be filters of $\mathbf{A}$.*

1. *If $\mathbf{A}$ satisfies $(\backslash\vee)$ and $\mathfrak{y}$ is prime, then $\mathfrak{x} \bullet \mathfrak{y} \in \mathcal{S}(A) \cup \{A\}$.*
2. *If $\mathbf{A}$ satisfies $(\vee/)$ and $\mathfrak{x}$ is prime, then $\mathfrak{x} \bullet \mathfrak{y} \in \mathcal{S}(A) \cup \{A\}$.*
3. *If $\mathbf{A}$ is a semilinear CRL, then $\mathfrak{x} \bullet \mathfrak{y} \in \mathcal{S}(A) \cup \{A\}$ provided that at least one of $\mathfrak{x} \in \mathcal{S}(A)$ or $\mathfrak{y} \in \mathcal{S}(A)$.*

Proof. To prove (1), note that $\mathfrak{x} \bullet \mathfrak{y}$ is a filter by Lemma 4.1.1(1). If $\mathfrak{y}$ is prime, let $x \vee y \in \mathfrak{x} \bullet \mathfrak{y}$. By definition there is then some $x' \in \mathfrak{x}$ and $y' \in \mathfrak{y}$ so that $x' \cdot y' \leqslant x \vee y$. This entails that $y' \leqslant x' \backslash (x \vee y)$, and applying $(\backslash\vee)$ gives $y' \leqslant (x' \backslash x) \vee (x' \backslash y)$, which is in $\mathfrak{y}$ because filters are up-sets. By the primality of $\mathfrak{y}$, one of $x' \backslash x$ or $x' \backslash y$ is in $\mathfrak{y}$. Hence one of $x' \cdot (x' \backslash x) \leqslant x$ or $x' \cdot (x' \backslash y) \leqslant y$ is in $\mathfrak{x} \bullet \mathfrak{y}$, whence $\mathfrak{x} \bullet \mathfrak{y}$ is prime or improper.

(2) follows from the a similar argument, and (3) follows because semilinear CRLs satisfy both $(\backslash\vee)$ and $(\vee/)$. □

Corollary 4.1.3. *Let* $\mathbf{A}$ *be a bounded semilinear CRL. Then* $\bullet$ *gives a partial binary operation on* $\mathcal{S}(A)$*, and is undefined exactly when* $\mathfrak{x} \bullet \mathfrak{y} = A$*. In particular, this claim holds if* $\mathbf{A} \in \mathsf{MTL} \cup \mathsf{SM}_{\perp}$*. If instead* $\mathbf{A} \in \mathsf{GMTL}$*, then* $\bullet$ *is a total operation on* $\mathcal{S}(A)$*.*

The previous results are phrased in terms of (generalized) prime filters, but we can also offer a treatment native to abstract spaces. Although we will only invoke this abstract description for MTL and GMTL, to state the result in full generality we let $\mathsf{sCRL}_{\perp}^{\tau}$ be the full subcategory of $\mathsf{RL}_{\perp}^{\tau}$ corresponding to semilinear bounded CRLs.

Lemma 4.1.4. *Let* $\mathbf{X} = (X, \leqslant, \tau, ^*, R, I)$ *be an object of* $\mathsf{sCRL}_{\perp}^{\tau}$*. If* $x, y, z \in X$ *satisfy* $R(x, y, z)$*, then there exists a least element* $z' \in X$ *such that* $R(x, y, z')$*. If* $\mathbf{X}$ *is an object of* GMTL^{τ}*, then for any* $x, y \in X$ *there exists a least* $z' \in X$ *with* $R(x, y, z')$*.*

Proof. According to extended Priestley duality, there exists a bounded semilinear CRL $\mathbf{A}$ so that $\mathbf{X} \cong \mathcal{S}(\mathbf{A})$. Let $\alpha \colon \mathbf{X} \to \mathcal{S}(\mathbf{A})$ be the map witnessing this isomorphism. Each of $\alpha(x)$, $\alpha(y)$, and $\alpha(z)$ are prime filters of $\mathbf{A}$, and moreover $R^{\mathcal{S}(\mathbf{A})}(\alpha(x), \alpha(y), \alpha(z))$. Thus $\alpha(x) \bullet \alpha(y) \subseteq \alpha(z)$.

Lemma 4.1.2 provides that $\alpha(x) \bullet \alpha(y)$ is either a prime filter of $\mathbf{A}$ or else coincides with A. Because $\alpha(z) \neq A$ and $\alpha(x) \bullet \alpha(y) \subseteq \alpha(z)$, the latter possibility cannot hold and thus $\alpha(x) \bullet \alpha(y) \in \mathcal{S}(A)$. Therefore $R^{\mathcal{S}(\mathbf{A})}(\alpha(x), \alpha(y), \alpha(x) \bullet \alpha(y))$. It follows that $R(x, y, \alpha^{-1}(\alpha(x) \bullet \alpha(y)))$ since α^{-1} is an isomorphism with respect to R. Also, if $z \in X$ and $R(x, y, z)$, then $\alpha(x) \bullet \alpha(y) \subseteq \alpha(z)$ by the isomorphism. Since α is an order isomorphism, we additionally have $\alpha^{-1}(\alpha(x)) \bullet \alpha(y)) \subseteq \alpha^{-1}(\alpha(z)) = z$. It follows that $z' := \alpha^{-1}(\alpha(x) \bullet \alpha(y))$ is the minimum element of $\{x \in X : R(x, y, z)\}$. This proves the claim for $\mathsf{sCRL}_{\perp}^{\tau}$.

The claim for GMTL^{τ} follows by the same argument, noting that in this setting if $x, y \in X$ then there always exists $z \in X$ with $R(x, y, z)$ as a consequence of $\bullet$ being total. $\square$

Given any object $\mathbf{X}$ of $\mathsf{sCRL}_{\perp}^{\tau}$ or GMTL^{τ}, the previous lemma permits us to define

$$x \bullet y = \begin{cases} \min\{z \in X : R(x, y, z)\}, & \text{if } \{z \in X : R(x, y, z)\} \neq \varnothing \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Of course, the second clause is unnecessary if $\mathbf{X}$ is in GMTL^{τ}.

Lemma 4.1.5. *Let $\mathbf{X}$ be an object of $\mathsf{sCRL}_{\perp}^{\tau}$ or GMTL^{τ}. Then each of the following holds in every instance where the occurrences of $\bullet$ are defined.*

1. *$R(x, y, z)$ iff $x \bullet y \leqslant z$.*
2. *$x \bullet (y \bullet z) = (x \bullet y) \bullet z$.*
3. *$x \bullet y = y \bullet x$.*
4. *If $x \leqslant y$, then $x \bullet z \leqslant y \bullet z$ and $z \bullet x \leqslant z \bullet y$.*

Proof. Note that if $R(x, y, z)$, then there is a least $z' \in X$ so that $R(x, y, z')$ by Lemma 4.1.4. We have that $z' = x \bullet y$ by definition, and therefore $x \bullet y \leqslant z$. On the other hand, if $x \bullet y$ is defined, then $R(x, y, x \bullet y)$ by the definition of $\bullet$. Moreover, if $x \bullet y \leqslant z$ then $R(x, y, z)$ since R is isotone in its third coordinate. This proves (1).

For the rest, let $\mathbf{A}$ be such that $\mathbf{X} \cong \mathcal{S}(\mathbf{A})$ and let $\alpha \colon \mathbf{X} \to \mathcal{S}(\mathbf{A})$ be an isomorphism. The proof of Lemma 4.1.4 demonstrates that $x \bullet y = \alpha^{-1}(\alpha(x) \bullet \alpha(y))$. As an immediate consequence, $\alpha(x \bullet y) = \alpha(x) \bullet \alpha(y)$. Filter multiplication is associative, commutative, and order-preserving for any CRL, so we obtain the result. □

The next proposition serves primarily to communicate some definitions in Chapter 8, but we again state it in more generality than necessary.

Proposition 4.1.6. *Let* $\mathbf{A}$ *be a bounded semilinear CRL or GMTL-algebra, and suppose that for* $\mathfrak{y}, \mathfrak{z} \in \mathcal{S}(A)$ *there exists* $\mathfrak{x} \in \mathcal{S}(A)$ *such that* $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$*. Then*

$$\max\{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}\}$$

exists. Moreover, this maximum is given by $\mathfrak{y} \Rightarrow \mathfrak{z}$*, where*

$$\mathfrak{y} \Rightarrow \mathfrak{z} := \bigcup\{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}\}.$$

Also, $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$ *if and only if* $\mathfrak{x} \subseteq \mathfrak{y} \Rightarrow \mathfrak{z}$*.*

Proof. We begin by observing that $\mathfrak{y} \Rightarrow \mathfrak{z}$ is a prime filter of $\mathbf{A}$. To see why, note that if $x \in \mathfrak{y} \Rightarrow \mathfrak{z}$ and $x \leqslant y$, then there is $\mathfrak{x} \in \mathcal{S}(A)$ such that $x \in \mathfrak{x}$ and $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. It follows that $y \in \mathfrak{x}$ because filters are up-sets, whence $y \in \mathfrak{y} \Rightarrow \mathfrak{z}$ and $\mathfrak{y} \Rightarrow \mathfrak{z}$ is an up-set.

To see that $\mathfrak{y} \Rightarrow \mathfrak{z}$ is close under $\wedge$, let $x, y \in \mathfrak{y} \Rightarrow \mathfrak{z}$. By definition there exist $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathcal{S}(A)$ such that $x \in \mathfrak{x}_1$, $y \in \mathfrak{x}_2$, $\mathfrak{x}_1 \bullet \mathfrak{y} \subseteq \mathfrak{z}$, and $\mathfrak{x}_2 \bullet \mathfrak{y} \subseteq \mathfrak{z}$. Let $\mathfrak{x}_1 \vee \mathfrak{x}_2$ be the filter generated by $\mathfrak{x}_1 \cup \mathfrak{x}_2$. Because filters are closed under $\wedge$, this gives $x \wedge y \in \mathfrak{x}_1 \vee \mathfrak{x}_2$.

We will prove that

$$(\mathfrak{x}_1 \vee \mathfrak{x}_2) \bullet \mathfrak{y} \subseteq \mathfrak{z}.$$

Pick $q \in (\mathfrak{x}_1 \vee \mathfrak{x}_2) \bullet \mathfrak{y}$. By definition there is $z \in \mathfrak{x}_1 \vee \mathfrak{x}_2$ and $w \in \mathfrak{y}$ satisfying $zw \leqslant q$. From the standard characterization of generated filters and the fact that $z \in \mathfrak{x}_1 \vee \mathfrak{x}_2$, we know that there are $z_1 \in \mathfrak{x}_1$, $z_2 \in \mathfrak{x}_2$ with $z_1 \wedge z_2 \leqslant z$. From the assumption we know $z_1 \cdot w \in \mathfrak{x}_1 \bullet \mathfrak{y} \subseteq \mathfrak{z}$ and $z_2 \cdot w \in \mathfrak{x}_2 \bullet \mathfrak{y} \subseteq \mathfrak{z}$, and by closure under $\wedge$ we get $(z_1 \cdot w) \wedge (z_2 \cdot w) \in \mathfrak{z}$. The distributive law $(\cdot\wedge)$ is satisfied in every semilinear CRL, whence $(z_1 \wedge z_2) \cdot w \in \mathfrak{z}$. This implies that $(z_1 \wedge z_2) \cdot w \leqslant z \cdot w \leqslant q$ is in $\mathfrak{z}$, whence $(\mathfrak{x}_1 \vee \mathfrak{x}_2) \bullet \mathfrak{y} \subseteq \mathfrak{z}$.

By Lemma 4.1.1(2), there exists a prime filter $\mathfrak{p}$ such that $\mathfrak{x}_1 \vee \mathfrak{x}_2 \subseteq \mathfrak{p}$ and $\mathfrak{p} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. Thus $x \wedge y \in \mathfrak{p}$ and $\mathfrak{p} \bullet \mathfrak{y} \subseteq \mathfrak{z}$, giving $x \wedge y \in \mathfrak{y} \Rightarrow \mathfrak{z}$. This suffices to prove that $\mathfrak{y} \Rightarrow \mathfrak{z}$ is a filter.

Next we prove that $\mathfrak{y} \Rightarrow \mathfrak{z}$ is prime, so pick $x \vee y \in \mathfrak{y} \Rightarrow \mathfrak{z}$. By definition there is $\mathfrak{x} \in \mathcal{S}(A)$ with $x \vee y \in \mathfrak{x}$ and $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. Because $\mathfrak{x}$ is prime, we know that $x \vee y \in \mathfrak{x}$ gives $x \in \mathfrak{x}$ or $y \in \mathfrak{x}$, whence $x \in \mathfrak{y} \Rightarrow \mathfrak{z}$ or $y \in \mathfrak{y} \Rightarrow \mathfrak{z}$. Additionally, notice that $\mathfrak{y} \Rightarrow \mathfrak{z} \subseteq \mathfrak{z}$ gives that $\mathfrak{y} \Rightarrow \mathfrak{z} \neq A$ provided that $\mathfrak{z} \neq A$. Therefore $\mathfrak{y} \Rightarrow \mathfrak{z} \in \mathcal{S}(\mathbf{A})$.

To prove the residuation property, first let $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{S}(A)$. Assume that $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. For every $x \in \mathfrak{x}$ we have $x \in \mathfrak{y} \Rightarrow \mathfrak{z}$ by definition, so $\mathfrak{x} \subseteq \mathfrak{y} \Rightarrow \mathfrak{z}$. Conversely, assume that $\mathfrak{x} \subseteq \mathfrak{y} \Rightarrow \mathfrak{z}$. Lemma 4.1.5 guarantees that $\bullet$ is order-preserving and commutative, so we have $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{y} \bullet (\mathfrak{y} \Rightarrow \mathfrak{z})$. Letting $z \in \mathfrak{y} \bullet (\mathfrak{y} \Rightarrow \mathfrak{z})$, by definition there exists $x \in \mathfrak{y}$ and $y \in \mathfrak{y} \Rightarrow \mathfrak{z}$ such that $xy \leqslant z$. From $y \in \mathfrak{y} \Rightarrow \mathfrak{z}$, we know that there exists $\mathfrak{w} \in \mathcal{S}(A)$ with $y \in \mathfrak{w}$ and $\mathfrak{w} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. Since $x \in \mathfrak{y}$ and $y \in \mathfrak{w}$, this gives $xy \in \mathfrak{w} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. Because $\mathfrak{z}$ is an up-set, this implies that $z \in \mathfrak{z}$, whence $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{y} \bullet (\mathfrak{y} \Rightarrow \mathfrak{z}) \subseteq \mathfrak{z}$. This suffices to show $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$ if and only if $\mathfrak{x} \subseteq \mathfrak{y} \Rightarrow \mathfrak{z}$, and that completes the proof. $\square$

By importing the above result to an abstract space by extended Priestley duality, we immediately obtain:

Corollary 4.1.7. *Let* $\mathbf{X}$ *be an object of* $\mathsf{sCRL}_{\perp}^{\tau}$ *or* GMTL^{τ}. *If for* $y, z \in X$ *there exists some* $x \in X$ *such that* $R(x, y, z)$, *then there is a least* $x' \in X$ *with* $R(x', y, z)$. *Moreover,* $x \bullet y \leqslant z$ *if and only if* $y \leqslant x'$.

We denote x' in the above by $y \Rightarrow z$.

We have seen that the (partial) prime filter operations $\bullet$ and $\Rightarrow$ may be defined on an abstract object $\mathbf{X}$ in $\mathsf{sCRL}_{\perp}^{\tau}$ (or in GMTL). Specializing to MTL, we give a similar analysis for the Routley star *.

Lemma 4.1.8. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, 1, 0)$ *be an MTL-algebra. For each* $\mathfrak{x} \in \mathcal{S}(A)$,

$$\mathfrak{x}^* = \max\{\mathfrak{y} \in \mathcal{S}(A) : \mathfrak{x} \bullet \mathfrak{y} \neq A\}.$$

In particular, the maximum above exists.

Proof. Note at the outset that $0 \notin \mathfrak{x} \bullet \mathfrak{x}^*$. To see this, suppose on the contrary that there exists $x \in \mathfrak{x}, y \in \mathfrak{x}^*$ with $xy \leqslant 0$. Then $y \leqslant x \rightarrow 0 = \neg x$. The prime filter $\mathfrak{x}^*$ is an up-set, so this gives $\neg x \in \mathfrak{x}^*$, and hence $\neg\neg x \notin \mathfrak{x}$. Because $\mathfrak{x}$ is an up-set and $x \leqslant \neg\neg x$, it follows that $x \notin \mathfrak{x}$. This is a contradiction, so $0 \notin \mathfrak{x} \bullet \mathfrak{x}^*$. It follows in particular that $\mathfrak{x} \bullet \mathfrak{x}^* \neq A$.

Next suppose that $\mathfrak{y} \nsubseteq \mathfrak{x}^*$. Then there is $y \in \mathfrak{y}$ with $y \notin \mathfrak{x}^*$, so $\neg y \in \mathfrak{x}$. This implies $\neg y \cdot y \in \mathfrak{x} \bullet \mathfrak{y}$. But $\neg y \cdot y = 0$ in any MTL-algebra, so $\mathfrak{x} \bullet \mathfrak{y} = A$. This proves the lemma. □

Corollary 4.1.9. *Let* $\mathbf{X}$ *be in* MTL^{τ}. *For each* $x \in X$, *there exists a greatest* $y \in X$ *so that there exists* $z \in X$ *with* $R(x, y, z)$. *Equivalently, there is a greatest* $y \in X$ *such that* $x \bullet y$ *is defined.*

Proof. From extended Priestley duality there is an MTL-algebra $\mathbf{A}$ with $\mathbf{X} \cong \mathcal{S}(\mathbf{A})$, and we let $\alpha\colon \mathbf{X} \to \mathcal{S}(\mathbf{A})$ be an isomorphism witnessing this fact. From Lemma 4.1.8 we know that $\alpha(x)^*$ is the greatest element of $\mathcal{S}(\mathbf{A})$ multiplying with $\alpha(x)$ to give a proper filter, and in particular $R^{\mathcal{S}(\mathbf{A})}(\alpha(x), \alpha(x)^*, \alpha(x) \bullet \alpha(x)^*)$. Using the fact that α^{-1} is an isomorphism, it follows that $R(x, \alpha^{-1}(\alpha(x)^*), \alpha^{-1}(\alpha(x) \bullet \alpha(x)^*))$.

Let $y \in X$, and suppose that there is $z \in X$ with $R(x, y, z)$. Then α being R-preserving gives $\alpha(x) \bullet \alpha(y) \subseteq \alpha(z)$. This implies that $\alpha(x) \bullet \alpha(y) \neq A$, and applying Lemma 4.1.8 yields $\alpha(y) \subseteq \alpha(x)^*$. Thus $y \leqslant \alpha^{-1}(\alpha(x)^*)$, proving that $\alpha^{-1}(\alpha(x)^*) = \max\{y \in X : (\exists z \in X)\ R(x, y, z)\}$ as desired. □

For an object $\mathbf{X}$ of MTL^τ and $x \in X$, define

$$x^* := \max\{y \in S : (\exists z \in X)\ R(x, y, z)\}.$$

This provides our abstract description of the Routley star.

4.2 Characterizing functionality

Section 4.1 reveals an unexpected connection between the functionality of extended Priestley duals and the distributive laws ($\backslash\vee$) and ($\vee/$) (see Section 2.1.1). The aim of this section is to achieve a deeper understanding of the role these distributive laws play in functionality. Our starting point is [32], where Gehrke explores the functionality phenomenon in order to understand topological algebras[12] as extended Priestley duals of certain residuated structures. [32] provides a second-order characterization of when extended Priestley duals are functional, but does not address

[12] A *topological algebra* of type σ is an algebra of type σ in the category of topological spaces. In other words, it is a topological space equipped with a continuous operation interpreting each function symbol in σ.

the role of the equational properties ($\backslash\vee$) and ($\vee/$). In order to do so, we recast Gehrke's results in the language of canonical extensions.

4.2.1 Residuation algebras and canonical extensions

The residuated structures in [32] are of a somewhat different kind than those introduced in Chapter 2. In order to conform with [32], for the purposes of this section we work with the algebraic structures defined as follows.

Definition 4.2.1 (cf. [32], Definition 3.14). *A* residuation algebra *is an algebra* $\mathbf{A} = (A, \wedge, \vee, \backslash, /, \bot, \top)$ *such that:*

1. $(A, \wedge, \vee, \bot, \top)$ *is a bounded distributive lattice.*

2. $\backslash$ *and* $/$ *are binary operations on* A *that preserve finite meets in their numerators.*

3. *For all* $x, y, z \in A$,

$$x \leqslant z/y \iff y \leqslant x\backslash z$$

As usual, the residuation law implies that $\backslash$ and $/$ convert joins in their denominators into meets.

Remark 4.2.2. Note that if $(A, \wedge, \vee, \cdot, \backslash, /, \bot, \top)$ is a distributive residuated binar (see Chapter 2), then $(A, \wedge, \vee, \backslash, /, \bot, \top)$ is a residuation algebra. In every residuation algebra with a complete lattice reduct, the residuals of $/$ and $\backslash$ may be defined as usual for complete residuated structures. In this case, item (3) of the previous definition entails that both $/$ and $\backslash$ share a common residual. The work to follow implies moreover that residuation algebras are exactly the multiplication-free subreducts of residuated binars.

Up to this point, we have worked with particular topological-relational representations of duals. For this section, we adopt a more abstract point of view and work in the setting of *canonical extensions*. A treatment of the theory of canonical extensions would take us far afield of our main purpose, but we recall a few of the main ideas. For more information on canonical extensions, see for example [29, Chapter 6] and [33].

Definition 4.2.3 ([33], Definition 1)**.** *Given any lattice* $\mathbf{L}$*, a* canonical extension *of* $\mathbf{L}$ *is a complete lattice* $\mathbf{L}^\delta$ *together with an embedding* $\mathbf{L} \hookrightarrow \mathbf{L}^\delta$ *satisfying:*

1. *Every element in* $\mathbf{L}^\delta$ *is a join of meets of elements of* $\mathbf{L}$ *and a meet of joins of elements of* $\mathbf{L}$ *(Density).*

2. *If* $A, B \subseteq L$ *and* $\bigwedge A \leqslant \bigvee B$ *in* $\mathbf{L}^\delta$*, then there are finite subsets* $A' \subseteq A$ *and* $B' \subseteq B$ *of with* $\bigwedge A' \leqslant \bigvee B'$ *(Compactness).*

Every lattice $\mathbf{L}$ has a canonical extension $\mathbf{L}^\delta$, and it is unique up to an isomorphism that fixes $\mathbf{L}$ (see, e.g., [33, Theorem 1]). We thus refer to $\mathbf{L}^\delta$ as *the* canonical extension of $\mathbf{L}$.

If $\mathbf{A} = (A, \wedge, \vee, \backslash, /, \bot, \top)$ is a residuation algebra, then the operations $\backslash, /, \bot, \top$ can be extended to the canonical extension of the lattice $(A, \wedge, \vee)$ (see [29, Section 6.1.2]) via the so-called π*-extensions* $\backslash^\pi$ and $/^\pi$ of the operations $\backslash$ and $/$, given by

$$x \backslash^\pi y := \bigvee \{x' \backslash y' : x', y' \in A \text{ and } x \leqslant x' \text{ and } y' \leqslant y\}$$

$$x /^\pi y := \bigvee \{x' / y' : x', y' \in A \text{ and } x' \leqslant x \text{ and } y \leqslant y'\}$$

When the canonical extension of $(A, \wedge, \vee)$ is endowed with these operations, it becomes a residuation algebra. We call the resulting residuation algebra the *canonical extension* of the residuation algebra $\mathbf{A}$, and denote it by $\mathbf{A}^\delta$. Because $\mathbf{A}^\delta$ has a complete lattice reduct, we may always define an operation $\cdot$ on $\mathbf{A}^\delta$ having $\backslash^\pi$ and $/^\pi$ as its residuals. We will freely make use of this operation when we work in the canonical extension of a residuation algebra. Note that by [15, Lemma 10.3.1], $\cdot$ restricts to the elements of the meet-closure of A in A^δ. For convenience, we denote the meet-closure of A in A^δ by $K(A^\delta)$, and the join-closure of A in A^δ by $O(A^\delta)$.

A variety of expanded lattices is called *canonical* if it is closed under taking canonical extensions, and an identity is called *canonical* if the variety it defines is canonical. The identities ($\backslash\vee$) and ($\vee/$) are canonical (see, for example, [29, Theorem 6.23]).

If $\mathbf{L}$ is a lattice, then recall that $x \in L$ is *completely join-irreducible* if for any subset $A \subseteq L$, $x = \bigvee A$ implies $x \in A$. The set of completely join-irreducible elements of $\mathbf{L}$ is denoted by $J^\infty(L)$. Note that whenever $\mathbf{L}$ is distributive, its canonical extension L^δ is completely distributive. This implies that each $x \in J^\infty(L^\delta)$ is *completely join-prime*; in other words, whenever $A \subseteq L^\delta$ with if $x \leqslant \bigvee A$ we have $x \leqslant a$ for some $a \in A$.

Canonical extensions play a role in duality theory because they provide an entirely algebraic means of understanding duals. In particular, if $\mathbf{L}$ is a distributive lattice, then $J^\infty(L^\delta)$ plays the same role as the poset of prime filters in Priestley duality. This leads us to the next definition

Definition 4.2.4. *Let* $\mathbf{A}$ *be a residuation algebra. Then the relational* dual structure *of* $\mathbf{A}$ *is* $\mathbf{A}_+^\delta := (J^\infty(A^\delta), \geqslant, R)$, *where* R *is a ternary relation on* $J^\infty(A^\delta)$ *defined for* $x, y, z \in J^\infty(A^\delta)$ *by*

$$R(x, y, z) \quad \textit{iff} \quad x \leqslant y \cdot z.$$

We say that the relation R *is* functional *if* $y \cdot z \in J^\infty(A^\delta) \cup \{\bot\}$ *when* $y, z \in J^\infty(A^\delta)$. *In this case, we also say that* $\mathbf{A}_+^\delta$ *is* functional.

We say that R is functional and defined everywhere *if $y \cdot z \in J^\infty(A^\delta)$ whenever $y, z \in J^\infty(A^\delta)$, in which case we say $\mathbf{A}_+^\delta$ is* total.

As a caution, note that the dual relation R is defined somewhat differently in the above than in previous chapters; we adopt this choice in order to conform with [26] (which itself follows [32]).

Note also that functional relations as defined in [32] coincide with relations that are functional and defined everywhere in Definition 4.2.4. The latter distinction is particularly important to us in light of the contrast between the extended Priestley dualities for MTL and GMTL (see Section 4.1) and the role of zero-divisors in that setting. Accordingly, we say that a residuation algebra $\mathbf{A}$ *extensionally has no zero-divisors* if $x \cdot y \neq \bot$ for all $x, y \in J^\infty(A^\delta)$.

4.2.2 The characterization

We have already seen several examples of residuation algebras whose duals are functional among semilinear CRLs (Section 4.1). In that setting, functionality is a consequence of the identities ($\backslash\vee$) or ($\vee/$). Because these are equational conditions, in that context we obtain the functionality of the dual of each algebra in an entire variety of residuated structures. The next example shows that this is atypical.

Example 4.2.5. *Let $\mathbb{Z}_3$ be the usual group of integers modulo 3. We consider its complex algebra $\mathbf{A} := (\mathcal{P}(\mathbb{Z}_3), \cap, \cup, \cdot, \backslash, /, \{0\})$, where the operations $\cdot$, $\backslash$, and $/$ are defined for $U, V \in \mathcal{P}(\mathbb{Z}_3)$ by*

$$U \cdot V := \{n + m : x \in U, y \in V\},$$

$$U \backslash V := \{k : U \cdot \{k\} \subseteq V\},$$

$$U/V := \{k : \{k\} \cdot V \subseteq U\}.$$

$\mathbf{A}$ *is a commutative residuated lattice, and* a fortiori *a residuation algebra. Because* $\mathbf{A}$ *is finite, we have* $\mathbf{A}^\delta = \mathbf{A}$*. Observe that for each* $n, m \in \mathbb{Z}_3$*, we have that*

$$\{n\} \cdot \{m\} = \{n + m\}.$$

Consequently, the dual relation R *on* $J^\infty(\mathcal{P}(\mathbb{Z}_3))$ *is functional and defined everywhere. This means that* $\mathbf{A}_+^\delta$ *is functional and total.*

Notice that $\{\varnothing, \{0\}, \{1,2\}, \mathbb{Z}_3\}$ *is the universe of a subalgebra of* $\mathbf{A}$*. In this subalgebra, we have* $\{1,2\} \cdot \{1,2\} = \mathbb{Z}_3$*, which is not join-irreducible despite the fact that* $\{1,2\}$ *is join-irreducible. Now since universal first-order sentences that are satisfied in some structure must also be satisfied in its substructures, this example illustrates that there is no universal first-order property in the language of residuated lattices that characterizes the functionality of the relational dual structure.*

Although we cannot offer a characterization of functionality in terms of universal sentences (much less equations), we will provide a second-order characterization. We begin with two technical lemmas that rephrase in the language of canonical extensions one of the key properties of prime filters (to wit, that each prime filter determines a maximal filter-ideal pair given by the prime filter and its complement).

Lemma 4.2.6. *Let* $\mathbf{L}$ *be a lattice. Suppose that* $k \in K(L^\delta)$ *is finitely prime and set* $o := \bigvee\{y \in L : y \not\geqslant k\}$*. Then* $k \not\leqslant o$*.*

Proof. Suppose on the contrary that $\bigwedge\{x \in L : k \leqslant x\} = k \leqslant o$. Compactness implies that there are finite sets $A \subseteq \{x \in L : k \leqslant x\}$ and $B \subseteq \{y \in L : y \not\geqslant k\}$ satisfying

$$x' := \bigwedge A \leqslant \bigvee B =: y'$$

This yields $x' \geqslant k$, and $y' \ngeqslant k$. To see why, note that if otherwise then the primality of k gives $y \geqslant k$ for some $y \in B$ (a contradiction). From this we obtain $k \leqslant x' \leqslant y'$, which contradicts $y' \ngeqslant k$. $\square$

Lemma 4.2.7. *Let* $\mathbf{L}$ *be a lattice. If* $k \in K(L^\delta)$ *is finitely prime, then* $k \in J^\infty(L^\delta)$.

Proof. By the density property of canonical extensions, it suffices to show that if $k = \bigvee A$ for some $A \subseteq K(L^\delta)$, then $k = a$ for some $a \in A$. Set

$$o := \bigvee\{x \in L : x \ngeqslant k\}.$$

We argue by contradiction, assuming $a < k$ for all $a \in A$. Observe that for every $a \in A$ we have

$$a = \bigwedge\{x \in L : x \geqslant a\}$$

as a consequence of $A \subseteq K(L^\delta)$. This implies that for each $a \in A$ there is $x_a \in L$ such that $x_a \geqslant a$ and $x_a \ngeqslant k$. Consequently, for each $a \in A$ we have $x_a \leqslant o$. This proves $\bigvee\{x_a : a \in A\} \leqslant o$, and thus

$$o \geqslant \bigvee\{x_a : a \in A\} \geqslant \bigvee A = k.$$

This is a contradiction to Lemma 4.2.6, and that settles the claim. $\square$

The above lemmas hold for an arbitrary lattice $\mathbf{L}$. The rest of the results of this section rely on the distributivity of the lattice reducts of residuation algebras.

Proposition 4.2.8. *Let* $\mathbf{A}$ *be a residuation algebra. If* $\mathbf{A} \models (\backslash\vee)$, *then* $\mathbf{A}^\delta_+$ *is functional.*

Proof. Because $(\backslash\vee)$ is canonical, the hypothesis gives that $\mathbf{A}^\delta \models (\backslash\vee)$. Let $x, y \in J^\infty(A^\delta)$ and suppose that $x \cdot y \neq \bot$. From $x, y \in J^\infty(A^\delta) \subseteq K(A^\delta)$, we have

$x \cdot y \in K(A^\delta)$ because $\cdot$ restricts to $K(A^\delta)$. From Lemma 4.2.7, it suffices to show that $x \cdot y$ is finitely prime.

Suppose that $x \cdot y \leqslant \bigvee S$ for a finite $S \subseteq A^\delta$. Residuating gives

$$y \leqslant x \backslash^\pi \bigvee S \leqslant \bigvee \{x \backslash^\pi s : s \in S\}$$

by $(\backslash \vee)$. Because the lattice reduct of $\mathbf{A}$ is distributive and y is prime, this gives $y \leqslant x \backslash s$ for some $s \in S$. Hence $x \cdot y \leqslant s$ for some $s \in S$, concluding the proof. $\square$

The following is an immediate consequence of Proposition 4.2.8.

Corollary 4.2.9. *Let* $\mathbf{A}$ *be a residuation algebra. If* $\mathbf{A}$ *satisfies* $(\backslash\vee)$ *and extensionally has no zero-divisors, then* $\mathbf{A}_+^\delta$ *is total.*

Remark 4.2.10. Although Proposition 4.2.8 and Corollary 4.2.9 address residuation algebras satisfying $(\backslash\vee)$, one may obtain the same results by entirely symmetric proofs if one replaces $(\backslash\vee)$ by $(\vee/)$.

Our last proposition of this chapter emends [32, Proposition 3.16], and provides our characterization of functionality on relational dual structures. (2) and (3) of Proposition 4.2.11 below reformulate (2) and (3) of [32, Proposition 3.16] in the language of canonical extensions. On the other hand, the condition given in Proposition 4.2.11(1) is weaker than that of [32, Proposition 3.16(1)]. In particular, it does not demand that the dual relation corresponding to $\cdot$ is defined everywhere. Although the proof of (1)$\Rightarrow$(2) is essentially that given in [32, Proposition 3.16], the proof of (3)$\Rightarrow$(1) is simpler than the corresponding proof in [32, Proposition 3.16], and is where the emendation occurs.

Proposition 4.2.11. *Let* $\mathbf{A} = (A, \wedge, \vee, /, \backslash, \bot, \top)$ *be a residuation algebra. The following are equivalent.*

1. $\mathbf{A}_+^\delta$ *is functional.*
2. *For all* $x, y, z \in A$ *and all* $j \in J^\infty(A^\delta)$*, if* $j \leqslant x$ *then there exists* $x' \in A$ *such that* $j \leqslant x'$ *and* $x\backslash(y \vee z) \leqslant (x'\backslash y) \vee (x'\backslash z)$.
3. *For all* $j \in J^\infty(A^\delta)$*, the map* $j\backslash^\pi(-) : O(A^\delta) \to O(A^\delta)$ *is* $\vee$*-preserving.*

Proof. To prove (1)$\Rightarrow$(2), let $x, y, z \in A$ and $j \in J^\infty(A^\delta)$ with $j \leqslant x$. Suppose that $k \in J^\infty(A^\delta)$ with $k \leqslant x\backslash(y \vee z)$. Then $x \cdot k \leqslant y \vee z$, so $j \cdot k \leqslant y \vee z$. By the hypothesis we have $j \cdot k \in J^\infty(A^\delta) \cup \{\bot\}$. Since completely join-irreducibles in a distributive lattice are prime, this implies that $j \cdot k \leqslant y$ or $j \cdot k \leqslant z$. Residuating, we obtain that one of

$$k \leqslant j \backslash^{\pi} y = \bigvee \{a \backslash y : a \in A \text{ and } j \leqslant a\}$$

$$k \leqslant j \backslash^{\pi} z = \bigvee \{a \backslash z : a \in A \text{ and } j \leqslant a\}$$

holds. Hence from $k \in J^{\infty}(A)$ we get that there is $x_k \in A$ such that $j \leqslant x_k$ and one of $k \leqslant x_k \backslash y$ or $k \leqslant x_k \backslash z$ holds. The latter fact gives $k \leqslant (x_k \backslash y) \vee (x_k \backslash z)$. Since $x_k \in A$ and $j \leqslant x_k$ for all such x_k, we get

$$x \backslash (y \vee z) = \bigvee \{k \in J^{\infty}(A) : k \leqslant x \backslash (y \vee z)\} \leqslant \bigvee \{(a \backslash y) \vee (a \backslash z) : a \in A \text{ and } j \leqslant x\}.$$

By compactness and because $\backslash$ is antitone in its denominator, there exist elements $a_1, \ldots, a_n \in A$ such that

$$x \backslash (y \vee z) \leqslant \bigvee \{(a_i \backslash y) \vee (a_i \backslash z) : 1 \leqslant i \leqslant n\} \leqslant (x' \backslash y) \vee (x' \backslash z)$$

where $x' := \bigwedge_{i=1}^{n} a_i \in A$. Because $j \leqslant x'$, this proves (1)⇒(2).

To prove (2)⇒(3), let $j \in J^{\infty}(A^{\delta})$ and $o_1, o_2 \in O(A^{\delta})$. Because $\backslash^{\pi}$ is isotone in its numerator, it suffices to show

$$x \backslash^{\pi} (o_1 \vee o_2) \leqslant (x \backslash^{\pi} o_2) \vee (x \backslash^{\pi} o_2). \tag{4.2.1}$$

The definition of the π-extension shows:

$$j \backslash^{\pi} (o_1 \vee o_2) = \bigvee \{x \backslash w : x, w \in A \text{ and } j \leqslant x \text{ and } w \leqslant o_1 \vee o_2\}$$

$$j \backslash^{\pi} o_1 = \bigvee \{x' \backslash y : x', y \in A \text{ and } j \leqslant x' \text{ and } y \leqslant o_1\}$$

$$j\backslash^{\pi} o_2 = \bigvee\{x'\backslash z : x', z \in A \text{ and } j \leqslant x' \text{ and } y \leqslant o_2\}$$

It suffices to show that for all $x, w \in A$ with $j \leqslant x$ and $w \leqslant o_1 \vee o_2$, there are $x', y, z \in A$ for which $j \leqslant x'$, $y \leqslant o_1$, $z \leqslant o_2$ and $x\backslash w \leqslant (x'\backslash y) \vee (x'\backslash z)$. We have $w \leqslant o_1 \vee o_2 = \bigvee\{y \in A : y \leqslant o_1\} \vee \bigvee\{z \in A : z \leqslant o_2\}$, and from compactness it follows that there exist $y, z \in A$ with $w \leqslant y \vee z$ and $y \leqslant o_1$, $z \leqslant o_2$. From the hypothesis, there exists $x' \in A$ so that $j \leqslant x'$ and $x\backslash w \leqslant x\backslash(y \vee z) \leqslant (x'\backslash y) \vee (x'\backslash z)$ as required.

To prove (3)⇒(1), let $j, k \in J^{\infty}(A^{\delta})$. Then $j \cdot k \in K(A^{\delta})$ because of general facts about canonical extensions of maps. From Lemma 4.2.7, it suffices to prove that if $x, y \in A^{\delta}$ and $j \cdot k \neq \perp$, then $j \cdot k \leqslant x \vee y$ implies $j \cdot k \leqslant x$ or $j \cdot k \leqslant y$. Density provides that it is enough to prove the claim for $x, y \in O(A^{\delta})$, and compactness provides that it is enough to prove the claim for $x \in A$ and $y \in A$. Note that if $j \cdot k \leqslant x \vee y$, then by residuation and the hypothesis we get $k \leqslant j\backslash^{\pi}(x \vee y) = (j\backslash^{\pi} x) \vee (j\backslash^{\pi} y)$. But k is prime, so $k \leqslant j\backslash^{\pi} x$ or $k \leqslant j\backslash^{\pi} y$. Hence $j \cdot k \leqslant x$ or $j \cdot k \leqslant y$ as needed. □

Chapter 5

Algebraic representations of Sugihara monoids

The array of duality-theoretic methods assembled in the foregoing chapters provides a toolkit for addressing algebraic questions, and now we begin deploying these tools. The next three chapters provide a duality-theoretic analysis of Sugihara monoids, and in particular of the equivalences of SM and $\mathsf{SM}_{\perp}$ to categories consisting of certain expansions of relative Stone algebras, first articulated in [30, 31]. Existing presentations of this equivalence are not amenable to our methods, so our task in the present chapter is to provide a more convenient rendition of these categorical equivalences. The version of the equivalence for $\mathsf{SM}_{\perp}$ obtained in this chapter provides the left-hand side of the diagram give in Figuer 1.1. After this algebraic preprocessing, Chapter 6 gives Esakia-like dualities for SM and $\mathsf{SM}_{\perp}$ via restriction of the Davey-Werner duality. Then Chapter 7 describes the equivalence for $\mathsf{SM}_{\perp}$ in terms of the relationship between this Esakia-like duality and the extended Priestley duality for $\mathsf{SM}_\perp$. The ideas in this chapter originally come from the author's [24].

5.1 The Galatos-Raftery construction

Recall that a *Sugihara monoid* is a distributive, idempotent, involutive CRL. A *relative Stone algebra* is a semilinear CRL for which $\cdot$ and $\wedge$ coincide, and a *Gödel algebra* is a bounded relative Stone algebra (see Section 2.3).

Definition 5.1.1. *Define EnSM^- to be the class of algebras $(A, \wedge, \vee, \to, e, N, f)$ satisfying the following.*

1. *$(A, \wedge, \vee, \to, e)$ is a relative Stone algebra.*
2. *$N \colon A \to A$ is a nucleus on $(A, \wedge, \vee, \to, e)$.*
3. *$f \in A$, and for all $a \in A$*
   - (a) $x \vee (x \to f) = e$
   - (b) $N(Na \to a) = e$
   - (c) *$Na = e$ if and only if $f \leqslant a$.*

Also define $\mathsf{EnSM}^-_\perp$ to be the class of expansions of members of EnSM^- by a designated least element.

Notwithstanding condition 3(c) in the previous definition, EnSM^- and $\mathsf{EnSM}^-_\perp$ are varieties (see [31]).

Let $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e, \neg)$ be a Sugihara monoid. Define

$$A^- := \{a \in A : a \leqslant e\}$$

and call the elements of A^- *negative*. The *enriched negative cone*[13] of $\mathbf{A}$ is the algebra

$$\mathbf{A}^- := (A^-, \wedge, \vee, \cdot, \to^-, e, N, \neg e),$$

where the operations $\to^-$ and N are defined by

$$a \to^- b := (a \to b) \wedge e$$

$$Na := (a \to e) \to e$$

For any Sugihara monoid $\mathbf{A}$, we have $\mathbf{A}^- \in \mathsf{EnSM}^-$. The analogous claim also holds for bounded Sugihara monoids and $\mathsf{EnSM}^-_\bot$, where we modify the definition of enriched negative cones to include a constant designating the least element. For both variants, the map $\mathbf{A} \mapsto \mathbf{A}^-$ becomes a functor by defining $h^- := h{\restriction}_{A^-}$ for a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$. The main result of [31] establishes that $(-)^-$ is one functor of a categorical equivalence between SM ($\mathsf{SM}_\bot$) and EnSM^- ($\mathsf{EnSM}^-_\bot$). The reverse functor produces a (bounded) Sugihara monoid from an arbitrary algebra in EnSM^- (respectively, $\mathsf{EnSM}^-_\bot$) by a process we call the *Galatos-Raftery construction*. This goes as follows. Let $\mathbf{A} = (A, \wedge, \vee, \to, e, N, f) \in \mathsf{EnSM}^-$. Define

$$\Sigma(A) = \{(a, b) \in A \times A : a \vee b = e \text{ and } Nb = b\},$$

and set $\Sigma(\mathbf{A}) := (\Sigma(\mathbf{A}), \sqcap, \sqcup, \circ, \Rightarrow, (e, e), \neg)$, where the operations are defined presently. Set

$$s := (a \to d) \wedge (c \to b)$$

[13] In the theory of residuated lattices, the notation $\mathbf{A}^-$ usually refers to the *negative cone* of $\mathbf{A}$, which coincides with the reducts of our enriched negative cones that are missing N and $\neg e$. We will not have occasion to refer to (unenriched) negative cones, so we repurpose this notation for our needs.

$$t := (a \to c) \wedge (d \to b)$$

and define

$$\begin{aligned}
(a,b) \sqcap (c,d) &= (a \wedge c, b \vee d) \\
(a,b) \sqcup (c,d) &= (a \vee c, b \wedge d) \\
(a,b) \circ (c,d) &= (s \to (a \wedge c), Ns) \\
(a,b) \Rightarrow (c,d) &= (t, N(t \to (a \wedge d))) \\
\neg(a,b) &= (a,b) \Rightarrow (f,e) \\
&= ((a \to f) \wedge b, N(((a \to f) \wedge b) \to a))
\end{aligned}$$

If $h\colon \mathbf{A} \to \mathbf{B}$ is a homomorphism between algebras in EnSM^-, define a morphism $\Sigma(h)\colon \Sigma(\mathbf{A}) \to \Sigma(\mathbf{B})$ by $\Sigma(h)(a,b) = (h(a), h(b))$. With this, Σ defines a functor from EnSM^- to SM. Moreover, the functor Σ can of modified to account for bounds: If $(\mathbf{A}, \bot)$ is an algebra in $\mathsf{EnSM}^-_\bot$, extend Σ by associating with $(\mathbf{A}, \bot)$ the Sugihara monoid $S(\mathbf{A})$ with designated least element $(\bot, t)$. Together with $(-)^-$, Σ gives a covariant equivalence of categories between SM and EnSM^- (as well as $\mathsf{SM}_\bot$ and $\mathsf{EnSM}^-_\bot$).

Σ is a variant of the twist product construction, which was first introduced by Kalman [41] in the context of normal distributive i-lattices (but twist products are now the subject of a considerable literature; see, e.g., [23, 42, 45, 46, 47, 54]). In Kalman's version of the construction, normal distributive i-lattices are represented as algebras built on a set of ordered pairs and the i-lattice involution is given by the operation $(a,b) \mapsto (b,a)$. Although the $(\wedge, \vee, \neg)$-reduct of any Sugihara monoid is a normal distributive i-lattice (see Proposition 2.3.4), observe that the involution $\neg$ in the definition of Σ is not given by $(a,b) \mapsto (b,a)$. On the other hand, for odd

Sugihara monoids the involution defined in Σ *is* given by $(a, b) \mapsto (b, a)$ (see [30], an antecedent of [31] for odd Sugihara monoids). This mismatch between the usual twist product involution and that given by Σ is unsuitable for our purposes, so we rephrase the construction outlined above in order to restore the natural involution $(a, b) \mapsto (b, a)$. Doing so demands further scrutiny of the algebraic structure EnSM^{-}, which we carry forth in the next section.

5.2 Algebras with Boolean constant

If $\mathbf{A}$ is a Brouwerian algebra and $\mathfrak{a}$ is a filter of $\mathbf{A}$, we say that $\mathfrak{a}$ is a *Boolean filter* of $\mathbf{A}$ if $\mathfrak{a}$ is a Boolean lattice[14] under the operations of $\mathbf{A}$. Note that $\{e\}$ is a Boolean filter for any Brouwerian algebra $\mathbf{A}$ with top element e.

Lemma 5.2.1. *Let $\mathbf{A} = (A, \wedge, \vee, \to, e)$ be a Brouwerian algebra, and let $\mathfrak{a}$ be a Boolean filter of $\mathbf{A}$ whose least element is f. Then for each $a \in \mathfrak{a}$, the complement of a in $\mathfrak{a}$ is $a \to f$.*

Proof. We have $a \to f \in \mathfrak{a}$ because $a \to f \geqslant f$. Since $a \in \mathfrak{a}$, this implies $a \wedge (a \to f) \in \mathfrak{a}$. From $a \wedge (a \to f) \leqslant f$ and f being the least element of $\mathfrak{a}$, we obtain that $a \wedge (a \to f) = f$. On the other hand, $\mathfrak{a}$ being a Boolean filter means that $a \in \mathfrak{a}$ has a complement c in $\mathfrak{a}$. This gives that $a \wedge c \leqslant f$, whence $c \leqslant a \to f$. Then $e = a \vee c \leqslant a \vee (a \to f)$, giving $a \vee (a \to f) = e$. This proves the result. $\square$

Proposition 5.2.2. *Let $\mathbf{A} = (A, \wedge, \vee, \to, e)$ be a Brouwerian algebra and let $f \in A$. Then the following are equivalent.*

1. *$a \vee (a \to f) = e$ for all $a \in {\uparrow} f$.*

2. *$a \vee (a \to f) = e$ for all $a \in A$.*

[14] In other words, $\mathfrak{a}$ is a complemented bounded distributive lattice. Of course, we do not assume that the lattice bounds are distinguished.

3. $\uparrow f$ is a Boolean lattice.

Proof. First, we show (1) implies (3). Suppose that $a \vee (a \to f) = e$ for all $a \in \uparrow f$ and let $a \in \uparrow f$. Then $a \wedge (a \to f) \leqslant f$. Since $a \to f \geqslant f$, we get $a, a \to f \in \uparrow f$. This yields $a \wedge (a \to f) = f$. Since $a \vee (a \to f) = e$ by assumption, this shows that each $a \in \uparrow f$ has a complement (i.e., $a \to f$) in $\uparrow f$, and hence that $\uparrow f$ is a Boolean filter.

Second, we show (3) implies (2). Suppose that $\uparrow f$ is a Boolean filter. Let $a \in A$. Then $a \to f \geqslant f$ gives $a \vee (a \to f) \in \uparrow f$. Thus $a \vee (a \to f)$ has a complement in $\uparrow f$, given by $(a \vee (a \to f)) \to f$ according to Lemma 5.2.1. Note that since $a \leqslant a \vee (a \to f)$ we get $(a \vee (a \to f)) \to f \leqslant a \to f$, so

$$\begin{aligned} e &= (a \vee (a \to f)) \vee ((a \vee (a \to f)) \to f) \\ &\leqslant a \vee (a \to f) \end{aligned}$$

This gives that $a \vee (a \to f) = e$ as claimed.

Since (2) implies (1) trivially holds, the result follows. □

Following Proposition 5.2.2, we say that an expansion of a Brouwerian algebra (Heyting algebra) **A** by a constant f satisfying $a \vee (a \to f) = e$ is a *Brouwerian algebra with Boolean constant* (respectively, *Heyting algebra with Boolean constant*). Our interest is in the semilinear members of these classes, and we denote the variety of relative Stone algebras with Boolean constant by bRSA and variety of Gödel algebras with Boolean constant by bGA. Algebras in these varieties are called *bRS-algebras* and *bG-algebras*, respectively.

We note that the comments on pp. 3207 and 3192 of [31] characterize the subdirect irreducibles in EnSM^{-} as follows.

Proposition 5.2.3. *$(A, \wedge, \vee, \rightarrow, e, N, f) \in \mathsf{EnSM}^-$ is subdirectly irreducible if and only if it is totally ordered, $\{a \in A : a < e\}$ has a greatest element, and one of the following holds:*

1. *$f = e$ and N is the identity function on A, or*
2. *f is the greatest element of $\{a \in A : a < e\}$, $Nf = e$, and $Na = a$ whenever $a \neq f$.*

The previous proposition leads us to the following important fact.

Lemma 5.2.4. *EnSM^- satisfies the identity $Na = f \rightarrow a$.*

Proof. It is enough to show that $Na = f \rightarrow a$ holds for subdirectly irreducibles, so let $\mathbf{A} = (A, \wedge, \vee, \rightarrow, e, N, f)$ be a subdirectly irreducible algebra in EnSM^-. There are two cases. First, if $f = e$ and N is the identity function on A, the result is trivial since $f \rightarrow a = e \rightarrow a = a = Na$ for every $a \in A$.

In the second case, $\mathbf{A}$ is a chain and N satisfies

$$Na = \begin{cases} e & a = f, e \\ a & a \neq f, e \end{cases}$$

Note that in any totally-ordered Brouwerian algebra,

$$x \rightarrow y = \begin{cases} e & x \leqslant y \\ y & x \nleqslant y \end{cases}$$

so

$$f \rightarrow a = \begin{cases} e & f \leqslant a \\ a & f \nleqslant a \end{cases}$$

Since e covers f in the second case, we get $f \leqslant a$ iff $a = f$ or $a = e$, proving the claim. $\square$

Proposition 5.2.5. *EnSM^- is term-equivalent to bRSA, and $\mathsf{EnSM}^-_\perp$ is term-equivalent to bGA.*

Proof. Lemma 5.2.4 shows that N is definable in the $(\wedge, \vee, \to, e, f)$-reduct of any $\mathbf{A} = (A, \wedge, \vee, \to, e, N, f) \in \mathsf{EnSM}^-$. The $(\wedge, \vee, \to, e, f)$-reduct of any such $\mathbf{A}$ satisfies $a \vee (a \to f) = e$ by definition, hence is a bRS-algebra.

Now suppose that $\mathbf{A} = (A, \wedge, \vee, \to, e, f)$ is a bRS-algebra. Define $N\colon A \to A$ by $Na = f \to a$. Then N is a nucleus from Example 2.3.5. Also, for any $a \in A$,

$$\begin{aligned} N(Na \to a) &= f \to ((f \to a) \to a) \\ &= (f \to a) \to (f \to a) \\ &= e \end{aligned}$$

whence we have the identity $N(Na \to a) = e$.

To see that we also have the condition that $Na = e$ if and only if $f \leqslant a$, observe

$$\begin{aligned} Na = e &\iff f \to a = e \\ &\iff e \leqslant f \to a \\ &\iff f \leqslant a. \end{aligned}$$

Thus every bRS-algebra is the $(\wedge, \vee, \to, e, f)$-reduct of some algebra in EnSM^-. It follows that EnSM^- is term-equivalent to bRSA, and the result for $\mathsf{EnSM}^-_\perp$ and bGA follows by an identical argument. $\square$

According to Proposition 5.2.5, we need not enrich the negative cones of Sugihara monoids by a nucleus in order to achieve categorical equivalence; the addition of a constant f satisfying $a \vee (a \to f) = e$ suffices. In particular, SM is categorically equivalent to bRSA. This equivalence is given as before, except with the following adjustments:

1. We modify the functor Σ by eliminating all occurrences of N in the definitions of $\circ$ and $\Rightarrow$ using the identity $Na = f \to a$.

2. We replace the functor $(-)^-$ with $(-)_{\bowtie} \colon \mathsf{SM} \to \mathsf{bRSA}$, defined for a Sugihara monoid $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e, \neg)$ by $\mathbf{A}_{\bowtie} = (A^-, \wedge, \vee, \to^-, e, \neg e)$.

Similar remarks apply to $\mathsf{SM}_\bot$ and bGA, which are equivalent by functors modified analogously to the above.

5.3 Naturalizing involution

The goal of this section is to replace Σ by an alternative functor $(-)^{\bowtie}$. Together with $(-)_{\bowtie}$, the functor $(-)^{\bowtie}$ yields an equivalence of categories between SM and bRSA (as well as between $\mathsf{SM}_\bot$ and bGA). However, $(-)^{\bowtie}$ yields a representation of Sugihara monoids that ties them more closely to their i-lattice reducts and hence to existing work on twist products.

For a bRS-algebra $\mathbf{A} = (A, \wedge, \vee, \to, e, f)$, define[15]

$$A^{\bowtie} = \{(a, b) \in A \times A : a \vee b = e \text{ and } a \wedge b \leqslant f\}$$

For $(a, b), (c, d) \in A \times A$, define

$$(a, b) \sqcap (c, d) = (a \wedge c, b \vee b)$$

[15] The notation $\bowtie$ comes from the theory of twist products. However, we caution that this is not to be confused with what is sometimes called in the literature the *full* twist product.

$$(a,b) \sqcup (c,d) = (a \vee c, b \wedge d)$$

as in the definition of Σ. Then $(A \times A, \sqcap, \sqcup)$ is a lattice (and in fact coincides with the product of the lattice reduct of $\mathbf{A}$ and its order dual).

Lemma 5.3.1. *Let* $\mathbf{A} = (A, \wedge, \vee, \rightarrow, e, f)$ *be a bRS-algebra. Then* $\Sigma(A)$ *and* $A^{\bowtie}$ *are universes of sublattices of* $(A \times A, \sqcap, \sqcup)$.

Proof. Let $(a,b),(c,d) \in A \times A$. First, suppose $a \vee b = c \vee d = e$. Then by the distributivity of the lattice reduct of $\mathbf{A}$,

$$\begin{aligned}(a \wedge c) \vee (b \vee d) &= ((a \vee b) \wedge (c \vee b)) \vee d \\ &= (e \wedge (c \vee b)) \vee d \\ &= e\end{aligned}$$

Similarly, $(a \vee c) \vee (b \wedge d) = e$.

Second, suppose $(a,b),(c,d) \in A \times A$ with $Nb = b$ and $Nd = d$, where $Nx = f \rightarrow x$ as above. Then $N(b \wedge d) = b \wedge d$ since $\mathbf{A}$ satisfies $(\backslash\wedge)$, and $N(b \vee d) = b \vee d$ since $\mathbf{A}$ satisfies $(\backslash\vee)$.

Third, suppose that $(a,b),(c,d) \in A \times A$ with $a \wedge b \leqslant f$ and $c \wedge d \leqslant f$. This gives

$$\begin{aligned}(a \wedge c) \wedge (b \vee d) &= (a \wedge c \wedge b) \vee (a \wedge c \wedge d) \\ &\leqslant (f \wedge c) \vee (f \wedge a) \\ &\leqslant f\end{aligned}$$

Similarly, $(a \vee c) \wedge (b \wedge d) \leqslant f$.

The first and second paragraphs above prove that $\Sigma(A)$ is closed under $\sqcap$ and $\sqcup$. The first and third paragraphs prove that $A^{\bowtie}$ is closed under $\sqcap$ and $\sqcup$. Hence both $\Sigma(A)$ and $A^{\bowtie}$ are universes of sublattices of $(A \times A, \sqcap, \sqcup)$ as claimed. $\square$

Given $\mathbf{A} = (A, \wedge, \vee, \rightarrow, e, f) \in \mathsf{bRSA}$, define $\overline{\delta}_{\mathbf{A}} \colon A \times A \rightarrow A \times A$ by

$$\overline{\delta}_{\mathbf{A}}(a, b) = (a, f \rightarrow b) = (a, Nb),$$

where $Nb = f \rightarrow b$ as usual.

Lemma 5.3.2. *$\overline{\delta}_{\mathbf{A}}$ is a lattice endomorphism of $(A \times A, \sqcap, \sqcup)$.*

Proof. Direct calculation using the identities $(\backslash\wedge)$ and $(\backslash\vee)$ shows

$$\overline{\delta}_{\mathbf{A}}((a, b) \sqcap (c, d)) = \overline{\delta}_{\mathbf{A}}(a, b) \sqcap \overline{\delta}_{\mathbf{A}}(c, d), \text{ and}$$

$$\overline{\delta}_{\mathbf{A}}((a, b) \sqcup (c, d)) = \overline{\delta}_{\mathbf{A}}(a, b) \sqcup \overline{\delta}_{\mathbf{A}}(c, d)$$

for any $(a, b), (c, d) \in A \times A$. $\square$

Suppose that $(a, b) \in A \times A$ satisfies $a \vee b = e$. The identity $f \rightarrow b \geqslant b$ implies that that $a \vee (f \rightarrow b) = e$. Also, the second coordinate of

$$\overline{\delta}_{\mathbf{A}}(a, b) = (a, f \rightarrow b) = (a, Nb)$$

is an N-fixed element of A. These remarks show that $\overline{\delta}_{\mathbf{A}}[A^{\bowtie}] \subseteq \Sigma(\mathbf{A})$, whence $\delta_{\mathbf{A}} \colon (A^{\bowtie}, \sqcap, \sqcup) \rightarrow (\Sigma(\mathbf{A}), \sqcap, \sqcup)$ defined by $\delta_{\mathbf{A}} = \overline{\delta}_{\mathbf{A}} \restriction_{A^{\bowtie}}$ is a lattice homomorphism.

Lemma 5.3.3. *$\delta_{\mathbf{A}}$ is a lattice isomorphism whose inverse is given by*

$$\delta_{\mathbf{A}}^{-1}(a, b) = (a, b \wedge (a \rightarrow f)).$$

Proof. It is enough to show that $\delta_{\mathbf{A}}$ is a bijection.

To see that $\delta_{\mathbf{A}}$ is one-to-one, let $(a,b),(c,d) \in A^{\bowtie}$ with $\delta_{\mathbf{A}}(a,b) = \delta_{\mathbf{A}}(c,d)$. Then $(a, f \to b) = (c, f \to d)$, i.e., $a = c$ and $f \to b = f \to d$. It follows that $f \to b \leqslant f \to d$, so $f \wedge b = f \wedge (f \to b) \leqslant d$. Because $(a,b) \in A^{\bowtie}$, we have $a \wedge b \leqslant f$ and $a \vee b = e$. By lattice distributivity, $(a \vee f) \wedge (b \vee f) = (a \wedge b) \vee f = f$. Also, $(a \vee f) \vee (b \vee f) = e \vee f = e$. It follows that $a \vee f$ and $b \vee f$ are complements in the Boolean lattice ${\uparrow}f$. Since $(a,d) \in A^{\bowtie}$ as well, an identical argument shows that $a \vee f$ and $d \vee f$ are complements in ${\uparrow}f$ too. But complements in a Boolean lattice are unique, whence $b \vee f = d \vee f$. Using $b \wedge f \leqslant d$ and distributivity, we obtain

$$\begin{aligned} b &= b \wedge (b \vee f) \\ &= b \wedge (d \vee f) \\ &= (b \wedge d) \vee (b \wedge f) \\ &\leqslant (b \wedge d) \vee d \\ &= d \end{aligned}$$

so that $b \leqslant d$. Similarly, we may prove $d \leqslant b$. It follows that $b = d$, and hence δ is one-to-one.

For proving that $\delta_{\mathbf{A}}$ is onto, let $(a,b) \in \Sigma(A)$. Then by definition $a \vee b = e$ and $b = f \to b$. Note that $a \wedge b \wedge (a \to f) = a \wedge f \wedge b \leqslant f$. Applying distributivity,

$$\begin{aligned} a \vee (b \wedge (a \to f)) &= (a \vee b) \wedge (a \vee (a \to f)) \\ &= e \vee e \\ &= e, \end{aligned}$$

whence $(a, b \wedge (a \to f)) \in A^{\bowtie}$. Moreover:

$$
\begin{aligned}
f \to (b \wedge (a \to f)) &= (f \to b) \wedge (f \to (a \to f)) \\
&= (f \to b) \wedge ((f \wedge a) \to f)) \\
&= (f \to b) \wedge e \\
&= f \to b \\
&= b
\end{aligned}
$$

This shows that $\delta_{\mathbf{A}}(a, b \wedge (a \to f)) = (a, b)$, and thus that $\delta_{\mathbf{A}}$ is onto. And the computation above actually proves more, viz. that the inverse of $\delta_{\mathbf{A}}$ is given by $(a, b) \mapsto (a, b \wedge (a \to f))$. $\square$

Since $(\Sigma(A), \sqcap, \sqcup)$ is the lattice reduct of the residuated lattice $\Sigma(\mathbf{A})$, we may transport structure along the lattice isomorphism $\delta_{\mathbf{A}}$ in order to equip $A^{\bowtie}$ with a residuated multiplication. By Lemma 5.3.3, $\delta_{\mathbf{A}}$ has an inverse $\delta_{\mathbf{A}}^{-1}$ defined by

$$\delta_{\mathbf{A}}^{-1}(a, b) = (a, b \wedge (a \to f)).$$

We define binary operations $\bullet$ and $\Rightarrow$ on $A^{\bowtie}$ by

$$
\begin{aligned}
(a, b) \bullet (c, d) &= \delta_{\mathbf{A}}^{-1}(\delta_{\mathbf{A}}(a, b) \circ \delta_{\mathbf{A}}(c, d)) \\
(a, b) \Rightarrow (c, d) &= \delta_{\mathbf{A}}^{-1}(\delta_{\mathbf{A}}(a, b) \Rightarrow \delta_{\mathbf{A}}(c, d))
\end{aligned}
$$

Unpacking this definition, $\bullet$ is given by $(a, b) \bullet (c, d) = (s, t)$, where

$$s = ((a \wedge f) \to d) \wedge [((c \wedge f) \to d) \to (a \wedge c)]$$

and

$$t = ((a \wedge f) \to d) \wedge ((c \wedge f) \to d) \wedge (s \to f).$$

Along the same lines, $\Rightarrow$ is given by $(a, b) \Rightarrow (c, d) = (w, v)$, where

$$w = (a \to c) \wedge ((f \wedge d) \to b)$$

and

$$v = [(f \wedge (a \to c) \wedge (d \to b)) \to (a \wedge (f \to d))] \wedge (w \to f).$$

By transport of structure, we immediately obtain:

Proposition 5.3.4. *Let* $\mathbf{A} = (A, \wedge, \vee, \to, e, f) \in \mathsf{bRSA}$. *Then the algebra*

$$(A^{\bowtie}, \sqcap, \sqcup, \bullet, \Rightarrow, (e, f))$$

is an idempotent, distributive CRL.

We may expand $(A^{\bowtie}, \sqcap, \sqcup, \bullet, \Rightarrow, (e, f))$ by the natural involution $\sim$ given by $\sim(a, b) = (b, a)$. Since $(a, b) \in A^{\bowtie}$ implies $(b, a) \in A^{\bowtie}$, $\sim$ is a well-defined binary operation on $A^{\bowtie}$. We will show that $(A^{\bowtie}, \sqcap, \sqcup, \bullet, \Rightarrow, (e, f)) \in \mathsf{SM}$. Toward this aim, we begin with a lemma.

Lemma 5.3.5. *If* $(a, b) \in A^{\bowtie}$, *then* $(a \to f) \wedge (f \to b) = b$.

Proof. Let $(a, b) \in A^{\bowtie}$, and note that by definition $a \wedge b \leqslant f$ and $a \vee b = e$. From $a \wedge b \leqslant f$ we have $b \leqslant a \to f$, whence $b = b \wedge (f \to b) \leqslant (a \to f) \wedge (f \to b)$. Also, Proposition 2.3.3 together with $a \vee b = e$ yields $a \to b = b$. Notice that $a \wedge (a \to f) \wedge (f \to b) \leqslant f \wedge (f \to b) \leqslant b$, so by applying the law of residuation $(a \to f) \wedge (f \to b) \leqslant a \to b = b$. This settles the claim. ∎

Proposition 5.3.6. *Let* $\mathbf{A} \in$ *bRSA. Then for all* $(a, b) \in A^{\bowtie}$, $\neg\delta_{\mathbf{A}}(a, b) = \delta_{\mathbf{A}}(\sim(a, b))$. *Thus* $\delta_{\mathbf{A}}$ *is an isomorphism of SM.*

Proof. Let $(a, b) \in A^{\bowtie}$. Then $a \vee b = e$, whence $a \to b = b$ and $b \to a = a$ by Proposition 2.3.3. Lemma 5.3.5 provides that $(a \to f) \wedge (f \to b) = b$. From these facts, we get

$$\begin{aligned}
\neg\delta_{\mathbf{A}}(a, b) &= \neg(a, f \to b) \\
&= (a, f \to b) \Rightarrow (f, e) \\
&= ((a \to f) \wedge (e \to (f \to b)), f \to [((a \to f) \wedge (e \to (f \to b)) \to (a \wedge e)]) \\
&= ((a \to f) \wedge (f \to b), f \to [((a \to f) \wedge (f \to b)) \to a) \\
&= (b, f \to (b \to a)) \\
&= (b, f \to a) \\
&= \delta_{\mathbf{A}}(\sim(a, b)).
\end{aligned}$$

This implies that $\delta_{\mathbf{A}}$ preserves $\sim$ as well as the other operations. It follows that $(A^{\bowtie}, \sqcap, \sqcup, \bullet, \Rightarrow, (e, f), \sim)$ is a Sugihara monoid that is isomorphic to $\Sigma(\mathbf{A})$ under $\delta_{\mathbf{A}}$ for every $\mathbf{A} \in$ bRSA. □

Define a functor $(-)^{\bowtie} \colon$ bRSA $\to$ SM as follows. If $\mathbf{A} = (A, \wedge, \vee, \to, e, f) \in$ bRSA then set

$$\mathbf{A}^{\bowtie} := (A^{\bowtie}, \sqcap, \sqcup, \bullet, \Rightarrow, (e, f), \sim).$$

For a homomorphism $h \colon \mathbf{A} \to \mathbf{B}$ of bRSA, define a function $h^{\bowtie} \colon \mathbf{A}^{\bowtie} \to \mathbf{B}^{\bowtie}$ by $h^{\bowtie}(a, b) = (h(a), h(b))$.

Lemma 5.3.7. *Let* $h \colon \mathbf{A} \to \mathbf{B}$ *be a morphism in bRSA. Then* $h^{\bowtie}$ *is a morphism in SM.*

Proof. Let $h\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism between bRS-algebras. From [31], $\Sigma(h)\colon \Sigma(\mathbf{A}) \to \Sigma(\mathbf{B})$ defined by $\Sigma(h)(a,b) = (h(a), h(b))$ is a homomorphism between Sugihara monoids. For any $(a,b) \in A^{\bowtie}$, we have

$$\begin{aligned}
\Sigma(h)(\delta_{\mathbf{A}}(a,b)) &= \Sigma(h)(a, f^{\mathbf{A}} \to b) \\
&= (h(a), h(f^{\mathbf{A}} \to b)) \\
&= (h(a), h(f^{\mathbf{A}}) \to h(b)) \\
&= (h(a), f^{\mathbf{B}} \to h(b)) \\
&= \delta_{\mathbf{B}}(h(a), h(b)) \\
&= \delta_{\mathbf{B}}(h^{\bowtie}(a,b)).
\end{aligned}$$

This demonstrates that $h^{\bowtie} = \delta_{\mathbf{B}}^{-1} \circ \Sigma(h) \circ \delta_{\mathbf{A}}$. The latter is a composition of morphisms in SM, which proves the claim. □

Lemma 5.3.8. *$(-)^{\bowtie}$ is functorial.*

Proof. Observe first that $(-)^{\bowtie}$ preserves the identity map. Let $g\colon \mathbf{A} \to \mathbf{B}$ and $h\colon \mathbf{B} \to \mathbf{C}$ be homomorphisms between bRS-algebras. Because Σ is a functor,

$$\begin{aligned}
(h \circ g)^{\bowtie} &= \delta_{\mathbf{C}}^{-1} \circ \Sigma(h \circ g) \circ \delta_{\mathbf{A}} \\
&= \delta_{\mathbf{C}}^{-1} \circ \Sigma(h) \circ \Sigma(g) \circ \delta_{\mathbf{A}} \\
&= \delta_{\mathbf{C}}^{-1} \circ \Sigma(h) \circ \delta_{\mathbf{B}} \circ \delta_{\mathbf{B}}^{-1} \circ \Sigma(g) \circ \delta_{\mathbf{A}} \\
&= h^{\bowtie} \circ g^{\bowtie}.
\end{aligned}$$

□

We have seen that $(-)^{\bowtie}\colon \mathsf{bRSA} \to \mathsf{SM}$ is a functor. We will show that it provides a reverse functor for $(-)_{\bowtie}$.

Lemma 5.3.9. *Let* $\mathbf{A} \in \mathsf{bRSA}$*. Then* $\mathbf{A} \cong (\mathbf{A}^{\bowtie})_{\bowtie}$*.*

Proof. We have $\mathbf{A}^{\bowtie} \cong \Sigma(\mathbf{A})$ under $\delta_{\mathbf{A}}$. From [31] we have $\Sigma(\mathbf{A})_{\bowtie} \cong \mathbf{A}$, and thus $(\mathbf{A}^{\bowtie})_{\bowtie} \cong \mathbf{A}$. ☐

Lemma 5.3.10. *Let* $\mathbf{A} \in \mathsf{SM}$*. Then* $\mathbf{A} \cong (\mathbf{A}_{\bowtie})^{\bowtie}$*.*

Proof. From [31] and $\delta_{\mathbf{A}_{\bowtie}}$, $\mathbf{A} \cong S(\mathbf{A}_{\bowtie}) \cong (\mathbf{A}_{\bowtie})^{\bowtie}$. ☐

Lemma 5.3.11. $\mathsf{bRSA}(\mathbf{A}, \mathbf{B})$ *and* $\mathsf{SM}(\mathbf{A}^{\bowtie}, \mathbf{B}^{\bowtie})$ *are in bijective correspondence.*

Proof. $\mathsf{bRSA}(\mathbf{A}, \mathbf{B})$ is in bijective correspondence with $\mathsf{SM}(\Sigma(\mathbf{A}), \Sigma(\mathbf{B}))$ by [31]. Also, for $h \colon \Sigma(\mathbf{A}) \to \Sigma(\mathbf{B})$, the map $h \mapsto \delta_{\mathbf{B}}^{-1} \circ h \circ \delta_{\mathbf{A}}$ gives a bijection between the $\mathsf{SM}(\Sigma(\mathbf{A}), \Sigma(\mathbf{B}))$ and $\mathsf{SM}(\mathbf{A}^{\bowtie}, \mathbf{B}^{\bowtie})$. This proves the lemma. ☐

Combining the results above:

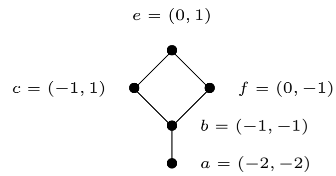
Theorem 5.3.12. $(-)^{\bowtie}$ *and* $(-)_{\bowtie}$ *give an equivalence of categories between* bRSA *and* SM*.*

The work above shows that $(-)^{\bowtie}$ and Σ are both adjoints of $(-)_{\bowtie}$. Consequently, $(-)^{\bowtie}$ and Σ are isomorphic functors. We therefore dispense with the functor Σ outright, and subsequently we will work exclusively with $(-)^{\bowtie}$ due to its more convenient involution. Of course, all of the above applies equally-well to *bounded* Sugihara monoids and bG-algebras.

Example 5.3.13. *Recall that we introduced the Sugihara monoid* $\mathbf{E}$ *in Example 2.3.11. The enriched negative cone of* $\mathbf{E}$ *is the bRS-algebra* $\mathbf{E}_{\bowtie}$ *with*

$$f = \neg(0,1) = (-0,-1) = (0,-1).$$

Its labeled Hasse diagram is



The nucleus $N\colon \mathbf{E}_{\bowtie} \to \mathbf{E}_{\bowtie}$ *defined by* $Nx = f \to x$ *is given by the equations*

$$Ne = Nf = e, \quad Nb = Nc = c, \quad Na = a.$$

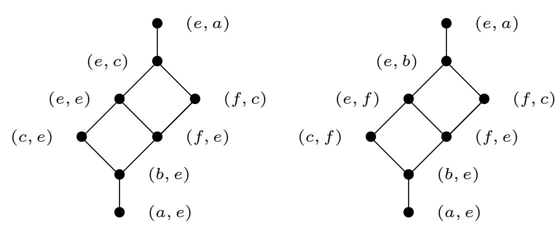
and thus

$$\begin{aligned}\Sigma(\mathbf{E}_{\bowtie}) &= \{(x, y) \in E^- \times E^- : x \vee y = e \textit{ and } Ny = y\} \\ &= \{(a, e), (b, e), (c, e), (f, e), (e, e), (e, a), (e, c), (f, c)\}.\end{aligned}$$

If we instead use $(-)^{\bowtie}$*, we get*

$$\begin{aligned}(\mathbf{E}_{\bowtie})^{\bowtie} &= \{(x, y) \in E^- \times E^- : x \vee y = e \textit{ and } x \wedge y \leqslant f\} \\ &= \{(a, e), (e, a), (b, e), (e, b), (e, f), (f, e), (f, c), (c, f)\}.\end{aligned}$$

The labeled Hasse diagrams for $\Sigma(\mathbf{E}_{\bowtie})$ *and* $(\mathbf{E}_{\bowtie})^{\bowtie}$ *are, respectively,*



Notice that $\Sigma(\mathbf{E}_{\bowtie})$ *and* $(\mathbf{E}_{\bowtie})^{\bowtie}$ *differ by only three pairs, including the monoid identity.*

Chapter 6

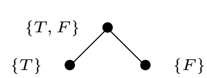
Esakia duality for Sugihara monoids

In [20], Dunn provides a relational semantics for the relevance logic R-mingle that employs a binary accessibility relation. In Dunn's terms, a *model structure* for R-mingle is a triple $(M, \bot, \leqslant)$, where M is a set, $\leqslant$ is a linear order on M, and $\bot \in M$ is the least element of M. If V is the collection of propositional variables over which the language of R-mingle is defined, a *model* for a model structure $(M, \leqslant, \bot)$ is a function $\alpha \colon V \times M \to \{\{T\}, \{F\}, \{T, F\}\}$ satisfying

(Heredity) If $x, y \in M$ and $x \leqslant y$, then $\alpha(p, x) \subseteq \alpha(p, y)$ for all $p \in V$.

After extending models to provide truth values in $\{\{T\}, \{F\}, \{T, F\}\}$ for complex sentences as well, Dunn defines a semantic consequence relation and shows that R-mingle is sound and complete with respect to this semantics.

Dunn's models for R-mingle are familiar: The heredity condition stipulates that the map $x \mapsto \alpha(p, x)$ is an isotone map from $(M, \leqslant)$ into the poset



which is nothing more than the poset reduct of the dualizing object $\underset{\sim}{\mathbf{D}}_3$ for NDIL (see Section 3.3). Sugihara monoids give the algebraic (rather than relational) semantics for R-mingle, and from Proposition 2.3.4 we know that they possess reducts in NDIL. We will see in this chapter that these connections to normal distributive i-lattices and their Davey-Werner duals manifests a duality for Sugihara monoids (as well as their bounded expansions). This duality is akin to Esakia duality, and we will obtain it by restricting the Davey-Werner duality in much the same way that Esakia duality is obtained by restricting Priestley duality. For $\mathsf{SM}_{\bot}$, this Esakia-style duality is rendered as the diagonal in Figuer 1.1.

Of course, Chapter 5 shows that SM ($\mathsf{SM}_{\bot}$) is equivalent to bRSA (bGA). The latter consists of expansions of certain Brouwerian algebras, which already enjoy the Esakia duality. It is natural to ask whether Esakia duality can be modified to account for the expansion by a Boolean constant. Constructing such a modification is our first order of business, and is the subject of Section 6.1. This modification of Esakia duality to account for the Boolean constant appears as the bottom of the diagram in Figuer 1.1. With this new variant of Esakia duality in hand, in Section 6.2 we will construct our duality for Sugihara monoids (with and without designated bounds) by restricting the Davey-Werner duality. Along the way, in Section 6.1.1 we will comment on the relationship between our variant of the Esakia duality for bGA and Bezhanishvili and Ghilardi's duality [4] for Heyting algebras expanded by nuclei. The content of this chapter is based on the author's [24].

6.1 Esakia duality for algebras with Boolean constant

We will show that bRSA is dually equivalent to the category of structured topological spaces defined in the following.

Definition 6.1.1. *A structure* $(X, \leqslant, D, \top, \tau)$ *is called a bRS-space if*

1. $(X, \leqslant, \top, \tau)$ *is a pointed Esakia space,*

2. $(X, \leqslant)$ *is a forest, and*

3. D *is a clopen subset of* X *consisting of designated* $\leqslant$*-minimal elements.*

For bRS-spaces $(X, \leqslant_X, D_X, \top_X, \tau_X)$ *and* $(Y, \leqslant_Y, D_Y, \top_Y, \tau_Y)$*, a function* α *from* $(X, \leqslant_X, D_X, \top_X, \tau_X)$ *to* $(Y, \leqslant_Y, D_Y, \top_Y, \tau_Y)$ *is called a* bRSS-morphism *if*

1. α *is a pointed Esakia map from* $(X, \leqslant_X, \top_X, \tau_X)$ *to* $(Y, \leqslant_Y, \top_Y, \tau_Y)$*,*

2. $\alpha[D_X] \subseteq D_Y$*, and*

3. $\alpha[D_X^{\mathsf{c}}] \subseteq D_Y^{\mathsf{c}}$.

We designate the category of bRS-spaces with bRSS*-morphisms by* bRSS*.*

As usual, to obtain a duality between bRSA and bRSS we will introduce new variants of $\mathcal{S}$ and $\mathcal{A}$. If $\mathbf{A} = (A, \wedge, \vee, \rightarrow, e, f)$ and $\mathbf{X} = (X, \leqslant, D, \top, \tau)$ are objects of bRSA and bRSS, respectively, define

$$\mathcal{S}(\mathbf{A}) = (\mathcal{S}(A, \wedge, \vee, \rightarrow, e), \varphi(f)^{\mathsf{c}})$$

$$\mathcal{A}(X, \leqslant, D, \top, \tau) = (\mathcal{A}(X, \leqslant, \top, \tau), D^{\mathsf{c}})$$

where $\mathcal{S}$ and $\mathcal{A}$ appearing on the right-hand sides of the above are their variants for Brouwerian algebras/pointed Esakia spaces. For morphisms, the definitions of $\mathcal{S}$ and $\mathcal{A}$ remain unmodified.

Lemma 6.1.2. *Let* $\mathbf{A} = (A, \wedge, \vee, \rightarrow, e, f)$ *be an object of* bRSA*. Then* $\mathcal{S}(\mathbf{A})$ *is an object of* bRSS*.*

Proof. The duality for Brouwerian algebras guarantees that $\mathcal{S}(\mathbf{A})$ is a pointed Esakia space whose underlying order is a forest. $\varphi(f)^{\mathsf{c}}$ is basic clopen, so it is enough to show that $\varphi(f)^{\mathsf{c}}$ consists of $\subseteq$-minimal elements. To see this, let $\mathfrak{y} \in \varphi(f)^{\mathsf{c}}$ and assume $\mathfrak{x} \in \mathcal{S}(A)$ with $\mathfrak{x} \subseteq \mathfrak{y}$. Let $y \in \mathfrak{y}$. Then $(y \to f) \vee y = e \in \mathfrak{x}$, so by the primality of $\mathfrak{x}$ either $y \in \mathfrak{x}$ or $y \to f \in \mathfrak{x}$. If $y \to f \in \mathfrak{x}$, then $y \to f \in \mathfrak{y}$. This gives $y \wedge (y \to f) \in \mathfrak{y}$. But $y \wedge (y \to f) \leqslant f$ and $\mathfrak{y}$ an up-set gives $f \in \mathfrak{y}$, which contradicts the choice of $\mathfrak{y}$. It follows that $y \in \mathfrak{x}$, so that $\mathfrak{y} \subseteq \mathfrak{x}$. Since $\mathfrak{x} \subseteq \mathfrak{y}$ as well, this shows that $\mathfrak{x} = \mathfrak{y}$ and thus $\mathfrak{y}$ is $\subseteq$-minimal. ☐

Lemma 6.1.3. *Let* $\mathbf{X} = (X, \leqslant, D, \top, \tau)$ *be an object of* bRSS*. Then* $\mathcal{A}(\mathbf{X})$ *is an object of* bRSA*.*

Proof. The duality for Brouwerian algebras guarantees that $\mathcal{A}(\mathbf{X})$ is a relative Stone algebra. We must show that D^{c} is a clopen up-set of $\mathbf{X}$, and that for any clopen up-set $U \subseteq X$ we have $U \cup (U \to D^{\mathsf{c}}) = X$. That D is clopen immediately implies that D^{c} is clopen, and that D consists of minimal elements immediately implies that D^{c} is an up-set.

For the rest, let $U \subseteq X$ be a clopen up-set and let $x \in X$. If $x \notin U$, then we claim that $x \in U \to D^{\mathsf{c}} = \{y \in X : {\uparrow}y \cap U \subseteq D^{\mathsf{c}}\}$. Let $y \in {\uparrow}x \cap U$. It suffices to show that y is not minimal. Note that $x \leqslant y$ and $y \in U$, so $x \notin U$ gives $x \neq y$. Thus y is not $\leqslant$-minimal, and hence $x \in U \to D^{\mathsf{c}}$. Thus $x \in U \cup (U \to D^{\mathsf{c}})$, so that $U \cup (U \to D^{\mathsf{c}}) = X$. ☐

Lemma 6.1.4. *Let* $h \colon \mathbf{A} \to \mathbf{B}$ *be a morphism of* bRSA*. Then* $\mathcal{S}(h) \colon \mathcal{S}(\mathbf{B}) \to \mathcal{S}(\mathbf{A})$ *is a morphism of* bRSS*.*

Proof. $\mathcal{S}(h)$ is a morphism of pointed Esakia spaces by the duality for Brouwerian algebras, so we need only show $\mathcal{S}(h)[\varphi(f^{\mathbf{B}})] \subseteq \varphi(f^{\mathbf{A}})$ and $\mathcal{S}(h)[\varphi(f^{\mathbf{B}})^{\mathsf{c}}] \subseteq \varphi(f^{\mathbf{A}})^{\mathsf{c}}$.

First, if $\mathfrak{x} \in \mathcal{S}(h)[\varphi(f^{\mathbf{B}})]$, then there is $\mathfrak{y} \in \varphi(f^{\mathbf{B}})$ such that $\mathfrak{x} = \mathcal{S}(h)(\mathfrak{y})$. As $h(f^{\mathbf{A}}) = f^{\mathbf{B}} \in \mathfrak{y}$, we have $f^{\mathbf{A}} \in h^{-1}[\mathfrak{y}] = \mathcal{S}(h)(\mathfrak{y}) = \mathfrak{x}$. Thus $\mathfrak{x} \in \varphi(f^{\mathbf{A}})$, whence $\mathcal{S}(h)[\varphi(f^{\mathbf{B}})] \subseteq \varphi(f^{\mathbf{A}})$.

Second, if $\mathfrak{x} \in \mathcal{S}(h)[\varphi(f^{\mathbf{B}})^{\mathsf{c}}]$, then there is $\mathfrak{y} \in \varphi(f^{\mathbf{B}})^{\mathsf{c}}$ so that $\mathfrak{x} = \mathcal{S}(h)(\mathfrak{y}) = h^{-1}[\mathfrak{y}]$. If $f^{\mathbf{A}} \in \mathfrak{x}$, then $f^{\mathbf{B}} = h(f^{\mathbf{A}})$ would give that $f^{\mathbf{B}} \in \mathfrak{y}$, contradicting $\mathfrak{y} \notin \varphi(f^{\mathbf{B}})$. Hence $f^{\mathbf{A}} \notin \mathfrak{x}$, yielding $\mathcal{S}(h)[\varphi(f^{\mathbf{B}})^{\mathsf{c}}] \subseteq \varphi(f^{\mathbf{A}})^{\mathsf{c}}$. □

Lemma 6.1.5. *Let $\alpha \colon \mathbf{X} \to \mathbf{Y}$ be a morphism of* bRSS*. Then $\mathcal{A}(\alpha) \colon \mathcal{A}(\mathbf{Y}) \to \mathcal{A}(\mathbf{X})$ is a morphism of* bRSA*.*

Proof. The duality for Brouwerian algebras shows that $\mathcal{A}(\alpha)$ is a morphism of BrA, so we need only show $\mathcal{A}(\alpha)(D_Y^{\mathsf{c}}) = D_X^{\mathsf{c}}$.

α being a bRSS-morphism gives that $\alpha[D_X] \subseteq D_Y$ and $\alpha[D_X^{\mathsf{c}}] \subseteq D_Y^{\mathsf{c}}$. From the latter, it follows that $D_X^{\mathsf{c}} \subseteq \alpha^{-1}[\alpha[D_X^{\mathsf{c}}]] \subseteq \alpha^{-1}[D_Y^{\mathsf{c}}]$. Hence $D_X^{\mathsf{c}} \subseteq \mathcal{A}(\alpha)(D_Y^{\mathsf{c}})$.

For the reverse inclusion, $D_X \subseteq \alpha^{-1}[\alpha[D_X]] \subseteq \alpha^{-1}[D_Y]$ follows from the other condition. Taking complements gives

$$D_X^{\mathsf{c}} \supseteq X - \alpha^{-1}[D_Y] = \alpha^{-1}[Y] - \alpha^{-1}[D_Y] = \alpha^{-1}[D_Y^{\mathsf{c}}] = \mathcal{A}(\alpha)(D_Y^{\mathsf{c}}).$$

This proves the claim. □

Lemma 6.1.6. *Let $\mathbf{A}$ be an object of* bRSA*. Then $\mathcal{AS}(\mathbf{A}) \cong \mathbf{A}$.*

Proof. Esakia duality gives that $\varphi \colon \mathbf{A} \to \mathcal{AS}(\mathbf{A})$ is an $(\wedge, \vee, \to, e)$-isomorphism of $\mathbf{A}$ with $\mathcal{AS}(\mathbf{A})$, so it is enough that φ preserves f. This follows from the equalities $f^{\mathcal{AS}(\mathbf{A})} = \mathcal{S}(A) - (\varphi(f^{\mathbf{A}})^{\mathsf{c}}) = \varphi(f^{\mathbf{A}})$. □

Lemma 6.1.7. *Let $\mathbf{X}$ and $\mathbf{Y}$ be objects of* bRSS*. If $\alpha \colon \mathbf{X} \to \mathbf{Y}$ is a* pEsa*-isomorphism, then α is an isomorphism of* bRSS *if and only if $\alpha[D_X] = D_Y$.*

Proof. For the forward direction, assume α is an isomorphism of bRSS. Then α has an inverse morphism in bRSS. From α being an isomorphism in pEsa we have that α is an isomorphism of posets, thus a bijection. Moreover, $\alpha[D_X] \subseteq D_Y$ and

$\alpha[D_X^{\mathsf{c}}] \subseteq D_Y^{\mathsf{c}}$. Because α is a bijection, taking complements in the latter inclusion gives $D_Y \subseteq \alpha[D_X^{\mathsf{c}}]^{\mathsf{c}} = \alpha[D_X]$, so $\alpha[D_X] = D_Y$.

For the backward implication, assume that $\alpha[D_X] = D_Y$. α being an isomorphism of pEsa gives that α is a bijection and its set-theoretic inverse α^{-1} coincides with its inverse in pEsa. That α is a bijection gives $\alpha[D_X^{\mathsf{c}}] = \alpha[D_X]^{\mathsf{c}} = D_Y^{\mathsf{c}}$. This shows that α is a morphism in bRSS. Also, $\alpha[D_X] = D_Y$ implies $\alpha^{-1}[D_Y] = D_X$ and $\alpha[D_X^{\mathsf{c}}] = D_Y^{\mathsf{c}}$ provides $\alpha^{-1}[D_Y^{\mathsf{c}}] = D_X^{\mathsf{c}}$. Hence α^{-1} is a morphism in bRSS. Therefore α is an isomorphism in bRSS, proving the result. □

Lemma 6.1.8. *Let* $\mathbf{X}$ *be an object of bRSS. Then* $\mathcal{SA}(\mathbf{X}) \cong \mathbf{X}$.

Proof. $\psi \colon \mathbf{X} \to \mathcal{SA}(\mathbf{X})$ is an isomorphism of pEsa by the duality for Brouwerian algebras. We aim to show that ψ is also an isomorphism of bRSS. Lemma 6.1.7 shows that it suffices to prove that $\psi[D] = \varphi(D^{\mathsf{c}})^{\mathsf{c}} = \{U \in \mathcal{A}(X) : D^{\mathsf{c}} \notin U\}$.

Let $\mathfrak{p} \in \psi[D]$. Then there exists $x \in D$ with $\mathfrak{p} = \psi(x)$, so $\mathfrak{p} = \{U \in \mathcal{A}(X) : x \in U\}$. From $x \notin D^{\mathsf{c}}$ we obtain $D^{\mathsf{c}} \notin \mathfrak{p}$, whence $\mathfrak{p} \in \varphi(D^{\mathsf{c}})^{\mathsf{c}}$. This shows $\psi[D] \subseteq \varphi(D^{\mathsf{c}})^{\mathsf{c}}$.

For the reverse inclusion, let $\mathfrak{p} \in \varphi(D^{\mathsf{c}})^{\mathsf{c}}$. Then $D^{\mathsf{c}} \notin \mathfrak{p}$. If $x \in D^{\mathsf{c}}$ such that $\psi(x) = \mathfrak{p}$, then $D^{\mathsf{c}} \in \{U \in \mathcal{A}(X) : x \in U\} = \psi(x) = \mathfrak{p}$. This is a contradiction, so $\mathfrak{p} \notin \psi[D^{\mathsf{c}}]$. As ψ is a bijection we have $\psi[D^{\mathsf{c}}] = \psi[D]^{\mathsf{c}}$ so that $\mathfrak{p} \notin \psi[D]^{\mathsf{c}}$, which provides $\mathfrak{p} \in \psi[D]$. Hence $\varphi(D^{\mathsf{c}})^{\mathsf{c}} \subseteq \psi[D]$, and it follows that $\psi[D] = \varphi(D^{\mathsf{c}})^{\mathsf{c}}$. □

Theorem 6.1.9. *bRSA and bRSS are dually-equivalent categories.*

Proof. This is immediately from Lemmas 6.1.2, 6.1.3, 6.1.4, 6.1.5, 6.1.6, and 6.1.8, noting that naturality follows from the proof that $\mathcal{S}$ and $\mathcal{A}$ give an equivalence between pEsa and the BrA. □

The foregoing work is phrased in terms of bRS-algebras, but the same analysis gives a duality for bG *mutatis mutandis*. The necessary modification amounts to dropping $\top$ from the signature of bRSS.

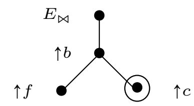


Figure 6.1: Labeled Hasse diagram for $\mathcal{S}(\mathbf{E}_{\bowtie})$

Definition 6.1.10. *A structure $(X, \leqslant, D, \tau)$ is called a bG-space if*

1. *$(X, \leqslant, \tau)$ is an Esakia space,*

2. *$(X, \leqslant)$ is a forest, and*

3. *D is a clopen subset of X consisting of $\leqslant$-minimal elements.*

For bG-spaces $\mathbf{X} = (X, \leqslant_X, D_X, \tau_X)$ and $\mathbf{Y} = (Y, \leqslant_Y, D_Y, \tau_Y)$, a map α from $\mathbf{X}$ to $\mathbf{Y}$ is called a bGS-morphism *if*

1. *φ is an Esakia map from $(X, \leqslant_X, \tau_X)$ to $(Y, \leqslant_Y, \tau_Y)$,*

2. *$\varphi[D_X] \subseteq D_Y$, and*

3. *$\varphi[D_X^{\mathsf{c}}] \subseteq D_Y^{\mathsf{c}}$.*

We designate the category of bG-spaces with bGS-morphisms by bGS.

Theorem 6.1.11. *bGA and bGS are dually-equivalent categories.*

Proof. The proof is identical to that of Theorem 6.1.9, except that we replace all references to Esakia duality for RSA by references to the Esakia duality for Gödel algebras. □

Example 6.1.12. *Recall the bRS-algebra $\mathbf{E}_{\bowtie}$ of Example 5.3.13. Its dual space is pictured in Figure 6.1. The elements of the designated subset are circled.*

6.1.1 bG-algebras as Heyting algebras with nuclei

Chapter 5 shows that bG-algebras are term-equivalent to their expansions by certain nuclei, viz. those given by $x \mapsto f \to x$. In [4], Bezhanishvili and Ghilardi introduced a duality for Heyting algebras equipped with nuclei, and in this section we compare our duality for bG-algebras with that of Bezhanishvili and Ghilardi. It turns out that the nucleus $x \mapsto f \to x$ of a bG-algebra presents itself in a very simply fashion on the dual space. Although nuclei are eliminable from the signature for our purposes, this nevertheless provides a different perspective for thinking about bG-spaces.

Definition 6.1.13. *A* nuclear Heyting algebra *is an algebraic structure of the form* $\mathbf{A} = (A, \wedge, \vee, \to, 1, 0, N)$*, where* $(A, \wedge, \vee, \to, 1, 0)$ *is a Heyting algebra and* N *is nucleus on* $(A, \wedge, \vee, \to, 1, 0)$*. We designate the category of nuclear Heyting algebras by* nHA*.*

Definition 6.1.14. *We call* $(X, \leqslant, R, \tau)$ *a* nuclear Esakia space *if* $(X, \leqslant, \tau)$ *is an Esakia space and* R *is a binary relation on* X *satisfying*

1. xRz *if and only if* $(\exists y \in X)(yRy$ *and* $x \leqslant y \leqslant z)$*,*
2. $R[x] = \{y \in X : xRy\}$ *is closed for each* $x \in X$*, and*
3. *whenever* $A \subseteq X$ *is clopen, so is* $R^{-1}[A] = \{x \in X : (\exists y \in A)\ xRy\}$*.*

A nuclear Esakia map *is an Esakia map* $\alpha \colon \mathbf{X} \to \mathbf{Y}$ *between nuclear Esakia spaces such that*

1. *if* $x, y \in X$ *with* $xR_{\mathbf{X}}y$*, then* $\alpha(x)R_{\mathbf{Y}}\alpha(x)$*, and*
2. *for all* $x \in X$ *and* $z \in Y$ *such that* $\alpha(x)R_{\mathbf{Y}}z$*, there exists* $y \in X$ *such that* $xR_{\mathbf{X}}y$ *and* $\alpha(y) = z$*.*

We denote the category of whose objects are nuclear Esakia spaces and whose morphisms are nuclear Esakia maps by nEsa*.*

We once again augment the functors $\mathcal{S}$ and $\mathcal{A}$. For a nuclear Heyting algebra $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 1, 0, N)$ and a nuclear Esakia space $\mathbf{X} = (X, \leqslant, R, \tau)$, define

$$\mathcal{S}(\mathbf{A}) = (\mathcal{S}(A, \wedge, \vee, \rightarrow, 1, 0), R_{\mathbf{A}})$$

$$\mathcal{A}(\mathbf{X}) = (\mathcal{A}(X, \leqslant, \tau), N_{\mathbf{X}})$$

where

- $R_{\mathbf{A}}$ is the binary relation on $\mathcal{S}(A)$ defined by $\mathfrak{x} R_{\mathbf{A}} \mathfrak{y}$ if and only if $N^{-1}[\mathfrak{x}] \subseteq \mathfrak{y}$,
- $N_{\mathbf{X}} \colon \mathcal{A}(X) \to \mathcal{A}(X)$ is defined by $N_{\mathbf{X}}(U) = X - R^{-1}[X - U]$

Define $\mathcal{S}$ and $\mathcal{A}$ on morphisms as usual. This set-up yields the following.

Theorem 6.1.15 ([4, Theorem 14])**.** *$\mathcal{S}$ and $\mathcal{A}$ give a dual equivalence of categories between* nHA *and* nEsa*.*

For each $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 1, 0, f) \in \mathsf{bGA}$, define $N_{\mathbf{A}} \colon A \to A$ by

$$N_{\mathbf{A}}(x) = f \to x.$$

Then $N_{\mathbf{A}}$ is a nucleus on $\mathbf{A}$, and $(A, \wedge, \vee, \rightarrow, 1, 0, N_{\mathbf{A}}) \in \mathsf{nHA}$. We will characterize the relation $R_{\mathbf{A}}$ corresponding to $N_{\mathbf{A}}$.

Given $\mathfrak{x} \in \mathcal{S}(A)$, set $\mathfrak{x}^{-1} := N_{\mathbf{A}}^{-1}[\mathfrak{x}]$ and observe that for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$,

$$\mathfrak{x} R_{\mathbf{A}} \mathfrak{y} \iff \mathfrak{x}^{-1} \subseteq \mathfrak{y}.$$

Lemma 6.1.16. *Let* $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 1, 0, f) \in \mathsf{bGA}$ *and let* $\mathfrak{x} \in \mathcal{S}(A)$*. Then* $\mathfrak{x}^{-1} \in \mathcal{S}(A) \cup \{A\}$*.*

Proof. Note that the laws ($\backslash\vee$) and ($\backslash\wedge$) give

$$N_{\mathbf{A}}(x \wedge y) = N_{\mathbf{A}}(x) \wedge N_{\mathbf{A}}(y) \text{ and } N_{\mathbf{A}}(x \vee y) = N_{\mathbf{A}}(x) \vee N_{\mathbf{A}}(y)$$

for all $x, y \in A$, whence $N_{\mathbf{A}}$ is a lattice homomorphism. The rest follows by noting that the inverse image of a prime filter under a lattice homomorphism must be prime or improper. $\square$

Remark 6.1.17. [4, Lemma 11] gives that $(-)^{-1}$ is a closure operator on the lattice of filters of $\mathbf{A}$. Combining this with Lemma 6.1.16, we obtain that $(-)^{-1}$ is a closure operator on $\mathcal{S}(A) \cup \{A\}$.

Lemma 6.1.18. *Let* $\mathbf{A} = (A, \wedge, \vee, \to, 1, 0, f) \in \mathsf{bGA}$ *and let* $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. *Then we have the following.*

1. *If* $\mathfrak{x}^{-1} \in \mathcal{S}(A)$, *then* $\mathfrak{x}^{-1}$ *is the least* $R_{\mathbf{A}}$*-successor of* $\mathfrak{x}$.
2. $\mathfrak{x} R_{\mathbf{A}} \mathfrak{x}$ *iff* $f \in \mathfrak{x}$.
3. *If* $\mathfrak{x}$ *is an* $R_{\mathbf{A}}$*-successor, then* $\mathfrak{x} R_{\mathbf{A}} \mathfrak{x}$.
4. *If* $\mathfrak{x} \subset \mathfrak{y}$, *then* $\mathfrak{x} R_{\mathbf{A}} \mathfrak{y}$.

Proof. To prove (1), suppose $\mathfrak{x}^{-1} \in \mathcal{S}(A)$. Since $\mathfrak{x}^{-1} \subseteq \mathfrak{x}^{-1}$, we have $\mathfrak{x} R_{\mathbf{A}} \mathfrak{x}^{-1}$. Now if $\mathfrak{y} \in \mathcal{S}(A)$ is an $R_{\mathbf{A}}$-successor of $\mathfrak{x}$, then $\mathfrak{x}^{-1} \subseteq \mathfrak{y}$ and $\mathfrak{x}^{-1}$ is the least $R_{\mathbf{A}}$-successor of $\mathfrak{x}$.

To prove (2), note that

$$N_{\mathbf{A}}(N_{\mathbf{A}}(x) \to x) = 1 \text{ and}$$

$$N_{\mathbf{A}}(x) = 1 \iff f \leqslant x,$$

whence $f \leqslant N_{\mathbf{A}}(x) \to x$ for all $x \in A$. This implies that $f \wedge N_{\mathbf{A}}(x) \leqslant x$ for each $x \in A$. Suppose $\mathfrak{x}$ is a filter with $f \in \mathfrak{x}$. Then for each $x \in \mathfrak{x}^{-1}$, we have $N_{\mathbf{A}}(x) \in \mathfrak{x}$. Since $\mathfrak{x}$ is a filter we have that $f \wedge N_{\mathbf{A}}(x) \in \mathfrak{x}$ also, whence $x \in \mathfrak{x}$. Thus $\mathfrak{x}^{-1} \subseteq \mathfrak{x}$, so $\mathfrak{x} R_{\mathbf{A}} \mathfrak{x}$. Conversely, if $\mathfrak{x} R_{\mathbf{A}} \mathfrak{x}$ then $\mathfrak{x}$ is an $R_{\mathbf{A}}$-successor of $\mathfrak{x}$. But $\mathfrak{x}^{-1}$ is the least $R_{\mathbf{A}}$-successor of $\mathfrak{x}$, so $\mathfrak{x}^{-1} \subseteq \mathfrak{x}$. Noting that $N_{\mathbf{A}}(f) = 1 \in \mathfrak{x}$, we have $f \in \mathfrak{x}^{-1}$ and hence $f \in \mathfrak{x}$.

To prove (3), suppose that $\mathfrak{y}$ is such that $\mathfrak{y} R_{\mathbf{A}} \mathfrak{x}$. Then $\mathfrak{y}^{-1} \subseteq \mathfrak{x}$. As $\mathfrak{y}^{-1} R_{\mathbf{A}} \mathfrak{y}^{-1}$, part (2) yields $f \in \mathfrak{y}^{-1}$ and hence $f \in \mathfrak{x}$. Therefore $\mathfrak{x} R_{\mathbf{A}} \mathfrak{x}$ by part (2).

To prove (4), let $\mathfrak{y} \in \mathcal{S}(A)$ with $\mathfrak{x} \subset \mathfrak{y}$. Then there is $x \in \mathfrak{y} - \mathfrak{x}$. From the definition of bG-algebras, $x \vee (x \to f) = 1$. Since $x \vee (x \to f) \in \mathfrak{x}$ and $\mathfrak{x}$ is prime with $x \notin \mathfrak{x}$, we get $x \to f \in \mathfrak{x}$. This provides $x, x \to f \in \mathfrak{y}$, and therefore $x \wedge (x \to f) \in \mathfrak{y}$. But $x \wedge (x \to f) \leqslant f$, and since $\mathfrak{y}$ is an up-set this implies $f \in \mathfrak{y}$. It follows from (2) that $\mathfrak{y} R_{\mathbf{A}} \mathfrak{y}$, whence $\mathfrak{y}^{-1} \subseteq \mathfrak{y}$. Since $\mathfrak{y} \subseteq \mathfrak{y}^{-1}$ always holds (i.e., since $(-)^{-1}$ is a closure operator), we get $\mathfrak{y}^{-1} = \mathfrak{y}$. From $\mathfrak{x} \subseteq \mathfrak{y}$ and $(-)^{-1}$ being isotone, we obtain $\mathfrak{x}^{-1} \subseteq \mathfrak{y}^{-1} = \mathfrak{y}$. Thus $\mathfrak{x} R_{\mathbf{A}} \mathfrak{y}$. $\square$

According to Lemma 6.1.18(4), only minimal elements of $\mathcal{S}(A)$ may fail to be reflexive under $R_{\mathbf{A}}$. From Definition 6.1.14(1), the accessibility relation of a nuclear Esakia space is determined by its order along with the collection of non-reflexive points. This motivates the following.

Definition 6.1.19. *Let $\mathbf{X} = (X, \leqslant, D, \tau)$ be a bG-space. Define a binary relation $\leqslant^{\sharp}_{\mathbf{X}}$ on X by*

$$\leqslant^{\sharp}_{\mathbf{X}} \; = \; \leqslant \cap \, \{(x,x) \in X \times X : x \in D\}^{\mathsf{c}}.$$

Call $\leqslant^{\sharp}_{\mathbf{X}}$ the sharp order *on $\mathbf{X}$.*

Proposition 6.1.20. *Let $\mathbf{A} = (A, \wedge, \vee, \to, 1, 0, f) \in \mathsf{bGA}$. Then $R_{\mathbf{A}}$ coincides with the sharp order on $\mathcal{S}(\mathbf{A})$.*

Proof. First, suppose $\mathfrak{x} R_{\mathbf{A}} \mathfrak{y}$. From Lemma 6.1.18(3), we have that $\mathfrak{y} R_{\mathbf{A}} \mathfrak{y}$ and from Lemma 6.1.18(2) it follows that $f \in \mathfrak{y}$. This entails that $\mathfrak{y} \in \varphi(f)$, whence

$$(\mathfrak{x}, \mathfrak{y}) \notin \{(\mathfrak{z}, \mathfrak{z}) \in \mathcal{S}(A) \times \mathcal{S}(A) : \mathfrak{z} \in \varphi(f)^{\mathsf{c}}\}.$$

Since $\mathfrak{x} \subseteq \mathfrak{y}$ follows from $\mathfrak{x} R_{\mathbf{A}} \mathfrak{y}$, this shows $\mathfrak{x} \leqslant^{\sharp}_{\mathcal{S}(\mathbf{A})} \mathfrak{y}$.

Conversely, suppose that $\mathfrak{x} \leqslant^{\sharp}_{\mathcal{S}(\mathbf{A})} \mathfrak{y}$. Then $\mathfrak{x} \subseteq \mathfrak{y}$ and

$$(\mathfrak{x}, \mathfrak{y}) \notin \{(\mathfrak{z}, \mathfrak{z}) : \mathfrak{z} \in \varphi(f)^{\mathsf{c}}\}.$$

We consider two cases. For the first case, suppose $\mathfrak{x} \neq \mathfrak{y}$. Then Lemma 6.1.18(4) implies that $\mathfrak{x} R_{\mathbf{A}} \mathfrak{y}$. For the second case, suppose that $\mathfrak{x} = \mathfrak{y} \notin \varphi(f)^{\mathsf{c}}$. Then $f \in \mathfrak{y}$, so $\mathfrak{y} R_{\mathbf{A}} \mathfrak{y}$ by Lemma 6.1.18(2). But since $\mathfrak{x} = \mathfrak{y}$, this gives $\mathfrak{x} R_{\mathbf{A}} \mathfrak{y}$. □

We now have a complete description of the accessibility relation arising from $N_{\mathbf{A}}$ for any given $\mathbf{A} \in \mathsf{bGA}$. The fact that together the order and $\varphi(f)^{\mathsf{c}}$ characterize $R_{\mathbf{A}}$ reflects the term-definability of $N_{\mathbf{A}}$ in the underlying bG-algebra (see Chapter 5), another aspect of which is recorded in the following.

Proposition 6.1.21. *Let $(X, \leqslant, D, \tau)$ be a bG-space. Then the image of X under $\leqslant^{\sharp}_{\mathbf{X}}$ is precisely D^{c}.*

Proof. First, let $y \in \leqslant^{\sharp}_{\mathbf{X}} [X]$. Then there is $x \in X$ with $x \leqslant^{\sharp}_{\mathbf{X}} y$. It follows that $x \leqslant y$ and one of $x \neq y$ or $x = y \notin D$ must hold. In the first case, y is not $\leqslant$-minimal and this gives $y \notin D$. In the second case, $y \notin D$ by hypothesis. This implies $y \notin D$ and $\leqslant^{\sharp}_{\mathbf{X}} [X] \subseteq D^{\mathsf{c}}$.

Second, let $y \in D^{\mathsf{c}}$. Then $y \leqslant y$ and $(y, y) \notin \{(x, x) : x \in D\}$, whence $y \leqslant^{\sharp}_{\mathbf{X}} y$. It follows that $y \in \leqslant^{\sharp} [X]$ and $D^{\mathsf{c}} \subseteq \leqslant^{\sharp} [X]$. Equality follows. □

Propositions 6.1.20 and 6.1.21 describe the relationship between the duality for bGA and the Bezhanishvili-Ghilardi duality for objects. What about morphisms? It turns out that not all nEsa-morphisms between objects of bGS are bGS-morphisms, but we obtain the appropriate morphisms if restrict our attention to nEsa-morphisms that preserve D.

Proposition 6.1.22. *Let $(X, \leqslant_X, D_X, \tau_X)$ and $(Y, \leqslant_Y, D_Y, \tau_Y)$ be bG-spaces and let $\alpha\colon X \to Y$ be a bGS-morphism. Then α is a nuclear Esakia map with respect to the relation $\leqslant^\sharp$.*

Proof. Note that α is an Esakia map by definition. We first show that α preserves $\leqslant^\sharp$, so let $x, y \in X$ with $x \leqslant^\sharp_{\mathbf{X}} y$. Then $x \leqslant_X y$, so $\alpha(x) \leqslant_Y \alpha(y)$ follows from α preserving $\leqslant$. Because $(x, y) \notin \{(z, z) : z \in D_X\}$, either $x \neq y$ or $x = y \notin D_X$. In the first case, $y \notin D_X$ since y is not minimal, hence as $\alpha[D_X^\mathsf{c}] \subseteq D_Y^\mathsf{c}$ we have $\alpha(y) \notin D_Y$. In the second case, if $x = y \notin D_X$ then $\alpha(y) \notin D_Y$ as well. This proves $(\alpha(x), \alpha(y)) \notin \{(z, z) : z \in D_Y\}$ in either case, so $\alpha(x) \leqslant^\sharp_{\mathbf{Y}} \alpha(y)$.

Second, let $x \in X$, $z \in Y$ such that $\alpha(x) \leqslant^\sharp_{\mathbf{Y}} z$. Then by definition

$$(\alpha(x), z) \notin \{(w, w) : w \in D_Y\},$$

and thus $\alpha(x) \neq z$ or $\alpha(x) = z \notin D_Y$. In the first case, $\alpha(x) \leqslant^\sharp_{\mathbf{Y}} z$ gives $\alpha(x) \leqslant_Y z$. Then since α is an Esakia map we have that there exists $y \in X$ with $x \leqslant y$ and $\alpha(y) = z$. From $\alpha(x) \neq z = \alpha(y)$, we infer $x \neq y$. Since $x \leqslant y$, this yields that y is not minimal, whence $y \notin D_X$. This implies that $x \leqslant^\sharp_{\mathbf{X}} y$ and $\alpha(y) = z$. In the second case, $\alpha(x) \notin D_Y$ and α preserving D_Y^c gives $x \notin D_X$, whence $x \leqslant^\sharp_{\mathbf{X}} x$ and $\alpha(x) = z$. This proves the result. □

Proposition 6.1.23. *Let $(X, \leqslant_X, D_X, \tau_X)$ and $(Y, \leqslant_Y, D_Y, \tau_Y)$ be bG-spaces and let $\alpha\colon X \to Y$ be an Esakia map that such that*

1. *for all* $x, y \in X$, $x \leqslant^{\sharp}_{\mathbf{X}} y$ *implies* $\alpha(x) \leqslant^{\sharp}_{\mathbf{Y}} \alpha(y)$,

2. *for all* $x \in X$ *and* $z \in Y$ *with* $\alpha(x) \leqslant^{\sharp}_{\mathbf{Y}} z$, *there exists* $y \in X$ *such that* $\alpha(y) = z$ *and* $x \leqslant^{\sharp}_{\mathbf{X}} y$, *and*

3. $\alpha[D_X] \subseteq D_Y$.

Then α *is a* bGS*-morphism.*

Proof. It is enough to show that $\alpha[D_X^{\mathsf{c}}] \subseteq D_Y^{\mathsf{c}}$, so let $y \in \alpha[D_X^{\mathsf{c}}]$. Then there is $x \in D_X^{\mathsf{c}}$ such that $\alpha(x) = y$. Since $x \in D_X^{\mathsf{c}}$ we have $x \leqslant^{\sharp}_{\mathbf{X}} x$, so $\alpha(x) \leqslant^{\sharp}_{\mathbf{Y}} \alpha(x)$. Thus $\alpha(x) \leqslant^{\sharp}_{\mathbf{Y}} y$, which entails that $y \in {\leqslant^{\sharp}_{\mathbf{Y}}}[Y] = D_Y^{\mathsf{c}}$ as desired. $\square$

Remark 6.1.24. We note that the term-equivalence of bGA to $\mathsf{EnSM}_{\bot}^{-}$ announced in Proposition 5.2.5 was originally discovered by applying the Bezhanishvili-Ghilardi duality. This provided valuable insight leading to the purely algebraic work of Chapter 5, which in turn supported the duality-theoretic innovations of this chapter. This offers a prime example of the mutually-supporting relationship between purely algebraic investigation and duality-theoretic study, as alluded to in Chapter 1.

6.2 Restricting the Davey-Werner duality

Proposition 2.3.4 gives that the $(\wedge, \vee, \neg)$-reduct of each Sugihara monoid is a normal distributive i-lattice, and an analogous statement holds for bounded Sugihara monoids and Kleene algebras. Let $\mathcal{U}\colon \mathsf{SM} \to \mathsf{NDIL}$ (or $\mathcal{U}\colon \mathsf{SM}_{\bot} \to \mathsf{KA}$) be the forgetful functor that associates to each (bounded) Sugihara monoid its reduct in NDIL (KA). Recalling that we denote the functors of the Davey-Werner duality by $\mathcal{D}$ and $\mathcal{E}$, the composite functor $\mathcal{DU}$ associates to each (bounded) Sugihara monoid the pointed Kleene space (respectively Kleene space) of its reduct. In order to simplify notation, we suppress $\mathcal{U}$ and simply write the Davey-Werner dual of (the appropriate reduct of) $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$ as $\mathcal{D}(\mathbf{A})$ (see Figuer 1.1). We also write $\mathcal{D}(A)$ for the carrier of $\mathcal{D}(\mathbf{A})$ as usual, and make note that for $\mathbf{A} \in \mathsf{SM}$ ($\mathsf{SM}_\perp$) we have that $\mathcal{D}(\mathbf{A})$ inherits its structure pointwise from $\underset{\sim}{\mathbf{D}}_3$ ($\underset{\sim}{\mathbf{K}}$). For clarity, we will write the order on $\mathcal{D}(\mathbf{A})$, $\underset{\sim}{\mathbf{D}}_3$, and $\underset{\sim}{\mathbf{K}}$ by $\lesssim$, and write the order on $\mathbf{D}_3$ and $\mathbf{K}$ by $\leqslant$. We will characterize subcategories of pKS and KS that are dually-equivalent via $\mathcal{D}$ to SM and $\mathsf{SM}_\perp$, respectively. To this end, we first identify the subcategories of interest and identify their connection to the dualities of the previous section.

6.2.1 Sugihara spaces

Definition 6.2.1. *We call a pointed Kleene space $(X, \leqslant, Q, D, \top, \tau)$ a* Sugihara space *if*

1. *$(X, \leqslant, \top, \tau)$ is a pointed Esakia space,*
2. *Q is the relation of comparability with respect to $\leqslant$ (in other words, set-theoretically $Q = \leqslant \cup \geqslant$), and*
3. *D is open.*

Since Q is comparability with respect to $\leqslant$, we typically suppress it and say that $(X, \leqslant, D, \top, \tau)$ is a Sugihara space.

Remark 6.2.2. Since D is closed in any pointed Kleene space, the condition that D is open in Definition 6.2.1 implies that D is clopen.

The following gives a connection to bRS-spaces.

Lemma 6.2.3. *Let $(X, \leqslant, D, \top, \tau)$ be a bRS-space. Then $(X, \leqslant, \leqslant \cup \geqslant, D, \top, \tau)$ is a Sugihara space.*

Proof. We first verify the conditions listed in Definition 3.3.3 for $(X, \leqslant, \top, \tau)$. Note that $(X, \leqslant, \top, \tau)$ is a pointed Esakia space with D clopen, and $Q = \leqslant \cup \geqslant$ is closed in X^2 since $\leqslant$ is closed in X^2 in any Priestley space. For the rest, let $Q = \leqslant \cup \geqslant$ be the relation of comparability with respect to $\leqslant$.

For (4)(a), $x\ Q\ x$ holds for each $x \in X$ since $x \leqslant x$.

For (4)(b), let $x, y \in X$ with $x\ Q\ y$ and $x \in D$. From $x\ Q\ y$ we have that $x \leqslant y$ or $y \leqslant x$. The former case gives $x \leqslant y$ immediately. If $y \leqslant x$, then the $\leqslant$-minimality of elements of D provides that $x = y$. Hence $x \leqslant y$ in either case.

For (4)(c), let $x, y, z \in X$ be such that $x\ Q\ y$ and $y \leqslant z$. Again $x\ Q\ y$ gives $x \leqslant y$ or $y \leqslant x$. In the first case, $x \leqslant y$ and $y \leqslant z$ gives $x \leqslant z$ by transitivity. In the second case, $y \leqslant x$ and $y \leqslant z$ gives $x, z \in {\uparrow}y$. The underlying poset $(X, \leqslant)$ of a bRS-space is a forest, so ${\uparrow}y$ is a chain and thus $x \leqslant z$ or $z \leqslant x$. This shows that $z\ Q\ x$, and thus the lemma. □

Lemma 6.2.4. *Let $(X, \leqslant, Q, D, \top, \tau)$ be a Sugihara space. Then $(X, \leqslant, D, \top, \tau)$ is a bRS-space.*

Proof. Definition 6.2.1 gives that $(X, \leqslant, \top, \tau)$ is a pointed Esakia space with D clopen, and it remains only to show that D consists of $\leqslant$-minimal elements and that $(X, \leqslant)$ is a forest.

To show that D consists of minimal elements, let $y \in D$ and suppose $x \leqslant y$. Because $x \leqslant y$ we get $y\ Q\ x$, whence $y\ Q\ x$ by Lemma 3.3.4(1). Since $y \in D$, this implies $y \leqslant x$ by Definition 3.3.3(4)(b). Since $x \leqslant y$, by antisymmetry $x = y$.

To show that $(X, \leqslant)$ is a forest, let $x \in X$ and let $y, z \in {\uparrow}x$. Observe that $x \leqslant y$ gives $y\ Q\ x$, and from $x \leqslant z$ and Definition 3.3.3(4)(c) we conclude $z\ Q\ y$. Then $z \leqslant y$ or $y \leqslant z$, which gives that ${\uparrow}x$ is a chain. □

bRS-spaces and Sugihara spaces are essentially the same objects according to Lemmas 6.2.3 and 6.2.4. However, they arise from entirely different duality-theoretic contexts: bRS-spaces are enriched Esakia duals of bRS-algebras, and Sugihara spaces are Davey-Werner duals of some normal distributive i-lattices. We will develop the connection between bRS-spaces and Sugihara spaces further, and exploit it to show that Sugihara spaces are duals of i-lattice reducts of Sugihara monoids.

Lemma 6.2.5. *Let* $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$ *and let* $h \in \mathcal{D}(A)$*. Then* $h^{-1}[\{0,1\}] \cap A^-$ *is a prime filter of* $\mathbf{A}_\bowtie$*.*

Proof. This is immediate because $\{0,1\}$ is a prime filter of each of $\mathbf{D}_3$ and $\mathbf{K}$, and h is a lattice homomorphism. □

For $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$, define $\xi_\mathbf{A} : \mathcal{D}(\mathbf{A}) \to \mathcal{S}(\mathbf{A}_\bowtie)$ by

$$\xi_\mathbf{A}(h) = h^{-1}[\{0,1\}] \cap A^-.$$

Note that $\xi_\mathbf{A}$ is well-defined from Lemma 6.2.5.

Lemma 6.2.6. *Let* $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$*. Then* $\xi_\mathbf{A}$ *is isotone.*

Proof. Let $h_1, h_2 \in \mathcal{D}(A)$ with $h_1 \lesssim h_2$. If $a \in \xi_\mathbf{A}(h_1)$, then $a \leqslant e$ and moreover $h_1(a) \in \{0,1\}$. Since $h_1 \lesssim h_2$, this gives $1 \lesssim h_1(a) \lesssim h_2(a)$. Therefore $a \in h_2^{-1}[\{0,1\}]$, whence $a \in \xi_\mathbf{A}(h_2)$. This shows $\xi_\mathbf{A}(h_1) \subseteq \xi_\mathbf{A}(h_2)$. □

Lemma 6.2.7. *Let* $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$ *and let* $h \in \mathcal{D}(A)$*. Then* $h(e) \in \{0,1\}$*.*

Proof. Note that $\neg e \leqslant e$ holds in $\mathbf{A}$. If $h(e) = -1$, then $h(\neg e) = \neg h(e) = 1$. But $\neg e \leqslant e$ gives $h(\neg e) \leqslant h(e)$, a contradiction since $1 \nleqslant -1$. The result follows. □

Lemma 6.2.8. *Let* $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$*. Then* $\xi_\mathbf{A}$ *is order-reflecting.*

Proof. Let $h_1, h_2 \in \mathcal{D}(A)$ such that $\xi_{\mathbf{A}}(h_1) \subseteq \xi_{\mathbf{A}}(h_2)$. If $h_1 \not\lesssim h_2$, then there exists $a \in A$ such that $h_1(a) \not\lesssim h_2(a)$. Then one of $h_2(a) = -1$ and $h_1(a) \neq -1$, or $h_2(a) = 1$ and $h_1(a) \neq 1$ must hold.

In the first case, $h_1(a) \in \{0, 1\}$ and from Lemma 6.2.7 it follows that

$$h_1(a \wedge e) = h_1(a) \wedge h_1(e) \in \{0, 1\}.$$

Since $a \wedge e \in A^-$, this implies $a \wedge e \in \xi_{\mathbf{A}}(h_1)$. Hence $a \wedge e \in \xi_{\mathbf{A}}(h_2)$. But $h_2(a) = -1$ and $h_2(e) \in \{0, 1\}$ implies $h_2(a \wedge e) = -1$, a contradiction.

In the second case, $h_1(a) \in \{-1, 0\}$ and $h_2(a) = 1$, so $h_1(\neg a) \in \{0, 1\}$ and $h_2(\neg a) = -1$. The second case therefore reduces to the first case, and we arrive at a contradiction again. It follows that $h_1 \lesssim h_2$, which proves the claim. $\square$

Lemma 6.2.9. *Let $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$. Then $\xi_{\mathbf{A}}$ is an order isomorphism.*

Proof. If is enough to show that $\xi_{\mathbf{A}}$ is surjective. For the case when $\mathbf{A} \in \mathsf{SM}$, observe that $h \colon A \to A$ defined by $h(a) = 0$ for all $a \in A$ is a $(\wedge, \vee, \neg)$-morphism such that $\xi_{\mathbf{A}}(h) = A^-$. Thus the improper filter is in the image of $\xi_{\mathbf{A}}$.

For the rest, let $\mathfrak{x}$ be a prime filter of $\mathbf{A}_{\bowtie}$. Because $\mathbf{A}$ has a distributive lattice reduct, $I = \{a \in A^- : a \notin \mathfrak{x}\}$ is a prime ideal of $\mathbf{A}_{\bowtie}$ since it is the complement of a prime filter. It is easy to see also that I is an ideal of $\mathbf{A}$. Moreover,

$$F = {\uparrow_{\mathbf{A}}}\mathfrak{x} = \{b \in A : a \leqslant b \text{ for some } a \in \mathfrak{x}\}$$

is a filter of $\mathbf{A}$, and $F \cap I = \varnothing$. The prime ideal theorem then asserts that there is a prime ideal J of $\mathbf{A}$ with $I \subseteq J$ and $F \cap J = \varnothing$. One may show that the set

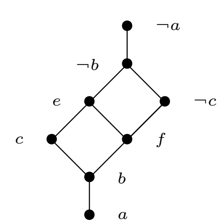
$\neg J = \{\neg a : a \in J\}$ is a prime filter of $\mathbf{A}$ as well. We define $h\colon A \to \{-1, 0, 1\}$ by:

$$h(a) = \begin{cases} 1 & \text{if } a \in \neg J \\ 0 & \text{if } a \notin J \cup \neg J \\ -1 & \text{if } a \in J \end{cases}$$

If $a, \neg a \in J$, then from the fact that J is an ideal we have $a \vee \neg a \in J$. The identity $e \leqslant a \vee \neg a$ holds in any Sugihara monoid, so J being a down-set implies $e \in J$. But this is impossible as $J \cap \mathfrak{x} = \varnothing$ and $e \in \mathfrak{x}$ (i.e., from $\mathfrak{x}$ being a prime filter of $\mathbf{A}_{\bowtie}$). The foregoing comments show that for each $a \in A$, either $a \notin J$ or $\neg a \notin J$, whence $J \cap \neg J = \varnothing$. Therefore at most one of $a \in \neg J$, $a \in J$, or $a \notin J \cup \neg J$ holds. At least one of $a \in J$, $a \in \neg J$, or $a \notin J \cup \neg J$ must hold as well, so h is a well-defined function.

It is a straightforward proof by cases to show that h is an i-lattice homomorphism, and must preserve the lattice bounds if they exist in $\mathbf{A}$. This shows that $h \in \mathcal{D}(A)$, and it is easy to see that $\xi_{\mathbf{A}}(h) = \mathfrak{x}$. It follows that $\xi_{\mathbf{A}}$ is surjective, and Lemmas 6.2.6 and 6.2.8 show that $\xi_{\mathbf{A}}$ is an order embedding. This suffices to settle the claim. $\square$

Example 6.2.10. *The algebra* $\mathbf{E}$ *from Example 2.3.11 has labeled Hasse diagram*



Consider the filter $\mathfrak{x} = \{b, c, f, e\}$ *of* E^-. *In the proof of Lemma 6.2.9, we have that* I *is* $\{a\}$, F *is* $A \backslash \{a\}$, J *is* $\{a\}$, *and* $\neg J$ *is* $\{\neg a\}$. *If* $\mathfrak{x} = \{c, e\}$ *instead, then* I *is* $\{a, b, f\}$, F *is* $\{c, e, \neg b, \neg a\}$, J *is* $\{a, b, f, \neg c\}$, *and* $\neg J$ *is* $\{c, e, \neg b, \neg a\}$. *For a final* *example, if* $\mathfrak{x} = \{e, f\}$*, then* I *is* $\{a, b, c\}$*,* F *is* $\{e, f, \neg b, \neg c, \neg a\}$*,* J *is* $\{a, b, c\}$*, and* $\neg J$ *is* $\{\neg c, \neg b, \neg a\}$*.*

Remark 6.2.11. Note that the partitions $\{J, (J \cup \neg J)^{\mathsf{c}}, \neg J\}$ provide a concrete rendering of the i-lattice homomorphisms into the i-lattice $\mathbf{D}_3$, just as prime filters provide a concrete rendering of morphisms into **2** in Priestley duality. Analogously, the pairs of clopens $\{U, V\}$ that determine the maps $C_{U,V}$ (see Section 3.3) provide a concrete representation of morphisms into $\underset{\sim}{\mathbf{D}}_3$ akin to how clopen up-sets provide a concrete representation of morphisms into the two-element linearly-ordered Priestley space. More will be said of such concrete representations in Section 6.3.

The following lemmas involve the topological structure of $\mathcal{D}(\mathbf{A})$, and we refer to the description of the subbasis on duals given in Lemma 3.3.10.

Lemma 6.2.12. *Let* $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$*. Then* $\xi_{\mathbf{A}}$ *is continuous.*

Proof. We show that inverse image under $\xi_{\mathbf{A}}$ of each subbasis element is open. Let $a \in A^-$. Then we have:

$$\begin{aligned}
\xi_{\mathbf{A}}^{-1}[\varphi(a)] &= \xi_{\mathbf{A}}^{-1}[\{\mathfrak{x} \in \mathcal{S}(A_{\bowtie}) : a \in \mathfrak{x}\}] \\
&= \{h \in \mathcal{D}(A) : a \in \xi_{\mathbf{A}}(h)\} \\
&= \{h \in \mathcal{D}(A) : a \in h^{-1}[\{0, 1\}] \cap A^-\} \\
&= \{h \in \mathcal{D}(A) : h(a) \in \{0, 1\}\} \\
&= \{h \in \mathcal{D}(A) : h(a) = 0\} \cup \{h \in \mathcal{D}(A) : h(a) = 1\} \\
&= U_{a,0} \cup U_{a,1}.
\end{aligned}$$

Thus $\xi_{\mathbf{A}}$ is continuous. □

Lemma 6.2.13. *Let* $\mathbf{A} \in \mathsf{SM}$ *(respectively,* $\mathsf{SM}_\perp$*). Then* $\mathcal{D}(\mathbf{A})$ *and* $\mathcal{S}(\mathbf{A}_{\bowtie})$ *are isomorphic in* Pries *(respectively,* pPries*).*

Proof. $\xi_{\mathbf{A}}$ is an order isomorphism from Lemma 6.2.9, and preserves $\top$ in the pointed case. This implies that $\xi_{\mathbf{A}}$ is a bijection. Since continuous bijections of compact Hausdorff spaces are homeomorphisms, Lemma 6.2.12 implies that $\xi_{\mathbf{A}}$ is a homeomorphism. Isomorphisms in Pries (respectively pPries) are (top-preserving) homeomorphisms that are order isomorphisms, so the result follows. □

As a consequence of the above, we obtain

Lemma 6.2.14. *Let $\mathbf{A} \in \mathsf{SM}$ (respectively, $\mathsf{SM}_\perp$). Then the (pointed) Priestley space reduct of $\mathcal{D}(\mathbf{A})$ is a (pointed) Esakia space.*

Proof. Every (pointed) Priestley space that is isomorphic to a (pointed) Esakia space is itself a (pointed) Esakia space. Thus Lemma 6.2.13 implies the result. □

Lemma 6.2.15. *Let $\mathbf{A} \in \mathsf{SM}$. If $\mathcal{D}(\mathbf{A}) = (\mathcal{D}(A), \lesssim, Q_{\mathbf{A}}, D, \top, \tau_{\mathbf{A}})$ is its Davey-Werner dual, then $(\mathcal{D}(A), \lesssim, D, \top, \tau_{\mathbf{A}})$ is a bRS-space. If instead $\mathbf{A} \in \mathsf{SM}_\perp$ and $\mathcal{D}(\mathbf{A}) = (\mathcal{D}(A), \lesssim, Q_{\mathbf{A}}, D, \tau_{\mathbf{A}})$ is its Davey-Werner dual, then $(\mathcal{D}(A), \lesssim, D, \tau_{\mathbf{A}})$ is a bG-space.*

Proof. Lemma 6.2.14 provides that $(\mathcal{D}(A), \lesssim, \top, \tau_{\mathbf{A}})$ is a pointed Esakia space. Since $\xi_{\mathbf{A}}$ is an order isomorphism, it follows from $(\mathcal{S}(A_{\bowtie}), \subseteq)$ being a forest that $(\mathcal{D}(A), \lesssim)$ is a forest as well. All that is left to show is that D is a clopen collection of $\lesssim$-minimal elements. That D consists of minimal elements follows from the fact $\mathcal{D}(\mathbf{A})$ is a pointed Kleene space. In order to prove that D is clopen, let $\mathfrak{x} = \xi_{\mathbf{A}}(h) = h^{-1}[\{0,1\}] \cap A^{-}$. For each $a \in \mathfrak{x}$, note that $h(a) \in \{0,1\}$ and

$$\begin{aligned} \mathfrak{x} \in \varphi(\neg e) &\iff \neg e \in \mathfrak{x} \\ &\iff h(\neg e) \in \{0,1\} \\ &\iff h(e) \in \{0,-1\}. \end{aligned}$$

Applying Lemma 6.2.7 then yields that $\mathfrak{x} \in \varphi(\neg e)$ if and only if $h(e) = 0$.

Note that by definition $h(a) \in \{-1, 1\}$ for all $h \in D$ and $a \in A$. It follows from this and the observation above that $\xi_{\mathbf{A}}(h) \notin \varphi(\neg e)$ for all $h \in D$, whence $\xi_{\mathbf{A}}[D] \subseteq \varphi(\neg e)^{\mathsf{c}}$. Moreover, if $\mathfrak{x} \in \varphi(\neg e)^{\mathsf{c}}$, then from the above we have $h(e) \notin \{0, -1\}$, whence $h(e) = 1$. Were it the case that $h(a) = 0$ for some $a \in A$, then $h(\neg a) = 0$ and we get $h(a \vee \neg a) = 0$. This is impossible since $e \leqslant a \vee \neg a$ and h is isotone, so the image of h is contained in $\{-1, 1\}$. Hence $\varphi(\neg e) \subseteq \xi_{\mathbf{A}}[D]$, and $\varphi(\neg e) = \xi_{\mathbf{A}}[D]$. Because $\xi_{\mathbf{A}}$ is a homeomorphism and $\varphi(\neg e)$ is clopen, we get that D is clopen as claimed.

The analogous result for $\mathbf{A} \in \mathsf{SM}_{\perp}$ follows similarly. $\square$

Lemma 6.2.16. *Let $\mathbf{A} \in \mathsf{SM}$. Then $\xi_{\mathbf{A}}$ is an isomorphism of bRS-spaces. If instead $\mathbf{A} \in \mathsf{SM}_{\perp}$, then $\xi_{\mathbf{A}}$ is an isomorphism of bG-spaces.*

Proof. Note that $\xi_{\mathbf{A}}$ is an isomorphism of pointed Priestley spaces by Lemma 6.2.13, and hence a pointed Esakia function. We show that $\xi_{\mathbf{A}}$ preserves the top element, the designated subset, and its complement. Observe that the map $\top \colon A \to \{-1, 0, 1\}$ defined by $\top(a) = 0$ is the greatest element of $\mathcal{D}(\mathbf{A})$, and

$$\xi_{\mathbf{A}}(\top) = \top^{-1}[\{0, 1\}] \cap A^{-} = A^{-}.$$

Since A^{-} is the $\subseteq$-greatest element of $\mathcal{S}(\mathbf{A}_{\bowtie})$, the top element is preserved.

To show that $\xi_{\mathbf{A}}$ preserves the designated subset and its complement, we show

$$\xi_{\mathbf{A}}[\{h \in \mathcal{D}(A) : (\forall a \in A)(h(a) \in \{-1, 1\})\}] = \varphi(\neg e)^{\mathsf{c}}.$$

To verify the forward inclusion, let $h \in \mathcal{D}(A)$ such that the image of h is contained in $\{-1, 1\}$. Since $h(e) \in \{0, 1\}$ this implies $h(e) = 1$, whence $h(\neg e) = -1$. Were it the case that $\xi_{\mathbf{A}}(h) \in \varphi(\neg e)$, this would imply $\neg e \in h^{-1}[\{0,1\}]$, a contradiction to $h(\neg e) = -1$. It follows that $\xi_{\mathbf{A}}(h) \in \varphi(\neg e)^{\mathsf{c}}$.

To verify the reverse inclusion, let $\mathfrak{x} \in \varphi(\neg e)^{\mathsf{c}}$ so that $\neg e \notin \mathfrak{x}$. By the surjectivity of $\xi_{\mathbf{A}}$, there exists $h \in \mathcal{D}(A)$ with $\xi_{\mathbf{A}}(h) = \mathfrak{x}$. Toward a contradiction, suppose that there is $a \in A$ such that $h(a) = 0$. The identities

$$x \wedge \neg x \leqslant \neg e \leqslant e \leqslant y \vee \neg y$$

hold in all Sugihara monoids, so in particular $a \wedge \neg a \leqslant \neg e \leqslant e \leqslant a \vee \neg a$. As $h(\neg a) = \neg h(a) = 0$, h being isotone provides

$$0 = h(a \wedge \neg a) \leqslant \neg e \leqslant e \leqslant h(a \vee \neg a) = 0,$$

This yields $h(\neg e) = h(e) = 0$, whence $\neg e \in h^{-1}[\{0,1\}] \cap A^{-} = \mathfrak{x}$. This contradicts $\neg e \notin \mathfrak{x}$, and therefore $h(a) \in \{-1,1\}$ for all $a \in A$. The reverse containment follows, and hence equality.

The above shows in particular that the designated subset is preserved by $\xi_{\mathbf{A}}$, and we only need show

$$\xi_{\mathbf{A}}[\{h \in A_{+} : (\exists a \in A)(h(a) = 0)\}] = \sigma(\neg t).$$

But this follows immediately by taking complements in the above since $\xi_{\mathbf{A}}$ is a bijection.

The case for $\mathbf{A} \in \mathsf{SM}_{\perp}$ follows analogously. $\square$

6.2.2 The duality

Section 6.2.1 lays the groundwork for connecting Sugihara monoids to Sugihara spaces by (1) demonstrating a close connection between $\mathcal{D}(\mathbf{A})$ and $\mathcal{S}(\mathbf{A}_{\bowtie})$ for any given (bounded) Sugihara monoid $\mathbf{A}$, and (2) developing the connection between bRS-spaces and Sugihara spaces. In this section, we tie the remaining threads together to provide our Esakia-style duality for Sugihara monoids.

Recall the functions $C_{U,V}$ were defined in Section 3.3 by

$$C_{U,V}(x) = \begin{cases} 1, & \text{if } x \notin V \\ 0, & \text{if } x \in U \cap V \\ -1, & \text{if } x \notin U \end{cases}$$

These functions completely characterize the morphisms $\mathbf{X} \to \underset{\sim}{\mathbf{D}}_3$ for any object $\mathbf{X}$ of pKS by Lemmas 3.3.5 and 3.3.6, and the same argument shows the analogous result in the $\top$-free setting. The following technical lemma demonstrates how to compute with the representation of normal distributive i-lattices afforded by the maps $C_{U,V}$ and the Davey-Werner duality.

Lemma 6.2.17. *Let $\underset{\sim}{\mathbf{L}} \in \{\underset{\sim}{\mathbf{D}}_3, \underset{\sim}{\mathbf{K}}\}$ and let $\alpha_1, \alpha_2 \colon \mathbf{X} \to \underset{\sim}{\mathbf{L}}$ be morphisms (in KS or pKS, as appropriate) with $\alpha_1 = C_{U_1,V_1}$ and $\alpha_2 = C_{U_2,V_2}$. Then*

1. $\neg C_{U_1,V_1} = C_{V_1,U_1}$.
2. $C_{U_1,V_1} \wedge C_{U_2,V_2} = C_{U_1 \cap U_2, V_1 \cup V_2}$, *and*
3. $C_{U_1,V_1} \vee C_{U_2,V_2} = C_{U_1 \cup U_2, V_1 \cap V_2}$,

Proof. To prove (1), note that for each $x \in X$ we have

$$\begin{aligned} \alpha_1(x) = 1 &\iff C_{U_1,V_1}(x) = 1 \\ &\iff x \notin V_1 \\ &\iff C_{V_1,U_1}(x) = -1. \end{aligned}$$

Similarly $\alpha_1(x) = -1$ if and only if $C_{V_1,U_1}(x) = 1$. Also, $\alpha_1(x) = 0$ if and only if $C_{V_1,U_1}(x) = 0$, whence $\neg\alpha_1 = C_{V_1,U_1}$.

To prove (2), note that in $\mathbf{L} \in \{\mathbf{D}_3, \mathbf{K}\}$ we have $a \wedge b = 1$ if and only if $a = 1$ and $b = 1$, and also $a \wedge b = -1$ if and only if $a = -1$ or $b = -1$. For each $x \in X$ we have

$$\begin{aligned} \alpha_1(x) \wedge \alpha_2(x) = 1 &\iff \alpha_1(x) = 1 \text{ and } \alpha_2(x) = 1 \\ &\iff C_{U_1,V_1}(x) = 1 \text{ and } C_{U_2,V_2}(x) = 1 \\ &\iff x \notin V_1 \text{ and } x \notin V_2 \\ &\iff x \notin V_1 \cup V_2 \\ &\iff C_{U_1 \cap U_2, V_1 \cup V_2}(x) = 1. \end{aligned}$$

By the same token,

$$\begin{aligned} \alpha_1(x) \wedge \alpha_2(x) = -1 &\iff \alpha_1(x) = -1 \text{ or } \alpha_2(x) = -1 \\ &\iff C_{U_1,V_1}(x) = -1 \text{ or } C_{U_2,V_2}(x) = -1 \\ &\iff x \notin U_1 \text{ or } x \notin U_2 \\ &\iff x \notin U_1 \cap U_2 \\ &\iff C_{U_1 \cap U_2, V_1 \cup V_2}(x) = -1. \end{aligned}$$

Similarly, $\alpha_1(x) \wedge \alpha_2(x) = 0$ if and only if $C_{U_1 \cap U_2, V_1 \cup V_2}(x) = 0$. Hence we obtain that $\alpha_1 \wedge \alpha_2 = C_{U_1 \cap U_2, V_1 \cup V_2}$.

(3) follows by a similar argument. $\square$

For each bRS-space $\mathbf{X}$, we define $\mu_{\mathbf{X}} \colon \mathcal{A}(\mathbf{X})^{\bowtie} \to \mathcal{E}(\mathbf{X}, \leqslant \cup \geqslant)$ by

$$\mu_{\mathbf{X}}(U, V) = C_{U,V}.$$

Note that for every $(U, V) \in \mathcal{A}(\mathbf{X})^{\bowtie}$ we have $U \cup V = X$ and $U \cap V \subseteq D^{\mathsf{c}}$. Also, for $(x, y) \in (X - U) \times (X - V)$ we have $x \notin U$ and $y \notin V$. Since $U \cup V = X$, this gives that $y \in U$ and $x \in V$. If $x \leqslant y$, then V being an up-set would give $y \in V$, a contradiction. Likewise, if $y \leqslant x$, then U being an up-set would give $x \in U$, again a contradiction. It follows that $[(X - U) \times (X - V)] \cap (\leqslant \cup \geqslant) = \varnothing$, and Lemma 3.3.5 provides that $\mu_{\mathbf{X}}$ is a well-defined pKS-morphism.

Lemma 6.2.18. *Let $\mathbf{A} \in \mathsf{bRSA}$. Then $\mathcal{E}(\mathcal{S}(\mathbf{A}), \subseteq \cup \supseteq)$ and $\mathbf{A}^{\bowtie}$ are isomorphic in* NDIL.

Proof. Note that $(\mathcal{S}(\mathbf{A}), \subseteq \cup \supseteq)$ is a pointed Kleene space by Lemma 6.2.3, whence $\mathcal{E}(\mathcal{S}(\mathbf{A}), \subseteq \cup \supseteq) \in \mathsf{NDIL}$. Lemma 6.1.6 gives $\mathcal{AS}(\mathbf{A})) \cong \mathbf{A}$ as bRS-algebras (and in particular as i-lattices), so it is enough show that $\mathcal{E}(\mathcal{S}(\mathbf{A}), \subseteq \cup \supseteq)$ is isomorphic as an i-lattice to $\mathcal{AS}(\mathbf{A})^{\bowtie}$. We will show $\mu = \mu_{\mathcal{S}(\mathbf{A})}$ is an i-lattice isomorphism.

Lemma 6.2.17 shows that μ is an i-lattice homomorphism from $\mathcal{AS}(\mathbf{A})^{\bowtie}$ to $\mathcal{E}(\mathcal{S}(\mathbf{A}), \subseteq \cup \supseteq)$, and Lemma 3.3.6 gives that μ is surjective. We will show that μ is one-to-one, so let $(U_1, V_1), (U_2, V_2) \in \mathcal{AS}(\mathbf{A})^{\bowtie}$ with $\mu(U_1, V_1) = \mu(U_2, V_2)$. Then

$C_{U_1,V_1} = C_{U_2,V_2}$, whence for all $x \in X$,

$$
\begin{aligned}
x \in U_1 &\iff C_{U_1,V_1}(x) \neq -1 \\
&\iff C_{U_2,V_2}(x) \neq -1 \\
&\iff x \in U_2
\end{aligned}
$$

Thus $U_1 = U_2$. One may likewise verify that $V_1 = V_2$, so $(U_1, V_1) = (U_2, V_2)$. Hence μ is an i-lattice isomorphism. □

We may now give our Esakia-style duality for Sugihara monoids. To do so, we define the appropriate morphisms.

Definition 6.2.19. *Let*

$$\mathbf{X} = (X, \leqslant_X, \leqslant_X \cup \geqslant_X, D_X, \top_X, \tau_X)$$

$$\mathbf{Y} = (Y, \leqslant_Y, \leqslant_Y \cup \geqslant_Y, D_Y, \top_Y, \tau_Y)$$

be Sugihara spaces. A bRSS*-morphism* α *from the bRS-space* $(X, \leqslant_X, D, \top_X, \tau_X)$ *to the bRS-space* $(Y, \leqslant_Y, D_Y, \top_Y, \tau_Y)$ *is said to be a* Sugihara space morphism*. We denote the category of Sugihara spaces with Sugihara space morphisms by* pSS*.*[16]

Remark 6.2.20. Each morphism of pSS is automatically a morphism of pKS despite the fact that the preservation of the relation $\leqslant \cup \geqslant$ is not explicitly demanded. This follows because a morphism always preserves the comparability relation when it preserves $\leqslant$.

We construct augmented variants of $\mathcal{D}$ and $\mathcal{E}$ as follows. Given $\mathbf{A} \in \mathsf{SM}$, let $\mathcal{D}(\mathbf{A})$ be the Davey-Werner dual of the i-lattice reduct of $\mathbf{A}$. Given a morphism $h\colon \mathbf{A} \to \mathbf{B}$ of SM, define $\mathcal{D}(h)\colon \mathcal{D}(\mathbf{B}) \to \mathcal{D}(\mathbf{A})$ by $h(x) = x \circ h$ as usual.

[16] Observe that here we include the leading p as a reminder that Sugihara spaces are top-bounded.

For the other functor, if $\mathbf{X} = (X, \leqslant, D, \top, \tau)$ is a Sugihara space we endow the Davey-Werner dual of $\mathbf{X}$ with additional binary operations $\cdot$ and $\to$ as follows. Given $\alpha_1 = C_{U_1,V_1}$ and $\alpha_2 = C_{U_2,V_2}$ maps in $\mathcal{E}(X)$, define

$$C_{(U_1,V_1)} \cdot C_{(U_2,V_2)} = \alpha_1 \cdot \alpha_2 = C_{(U_1,V_1)\bullet(U_2,V_2)}$$

$$C_{(U_1,V_1)} \to C_{(U_2,V_2)} = \alpha_1 \to \alpha_2 = C_{(U_1,V_1)\Rightarrow(U_2,V_2)},$$

where $\bullet$ and $\Rightarrow$ are the operations on the Sugihara monoid $\mathcal{A}(\mathbf{X})^{\bowtie}$ (see Section 5.3). If $\wedge$, $\vee$, and $\neg$ are the operations of the Davey-Werner dual of $\mathbf{X}$, we set

$$\mathcal{E}(\mathbf{X}) = (\mathcal{E}(X), \wedge, \vee, \cdot, \to, C_{X,D^c}, \neg),$$

where $\mathcal{E}(X)$ denotes the collection of pKS-morphisms $\mathbf{X} \to \underset{\sim}{\mathbf{D}}_3$ as usual. Given a morphism $\alpha\colon \mathbf{X} \to \mathbf{Y}$ of pSS, define $\mathcal{E}(\alpha)\colon \mathcal{E}(\mathbf{Y}) \to \mathcal{E}(\mathbf{X})$ by $\mathcal{E}(\alpha)(\beta) = \beta \circ \alpha$ as usual.

Remark 6.2.21. Note that the above augmentations make the map $\mu_{\mathbf{X}}$ into a Sugihara monoid isomorphism by construction.

Lemma 6.2.22. *Let $\mathbf{A} \in \mathsf{SM}$. Then $\mathcal{E}(\mathbf{A})$ is a Sugihara space.*

Proof. Let $(\mathcal{D}(A), \lesssim, Q_{\mathbf{A}}, D, \top, \tau_{\mathbf{A}})$ be the Davey-Werner dual of the i-lattice reduct of $\mathbf{A}$. Lemma 6.2.15 gives that $(\mathcal{D}(A), \lesssim, D, \top, \tau_{\mathbf{A}})$ is a bRS-space. From Lemma 6.2.3 it is enough to show that the relation $Q_{\mathbf{A}}$ is $\lesssim$-comparability.

The Davey-Werner duality provides that $\mathcal{E}\mathcal{D}(\mathbf{A})$ and $\mathbf{A}$ are isomorphic as lattices with involution. Since $(\mathbf{A}_{\bowtie})^{\bowtie}$ and $\mathbf{A}$ are isomorphic Sugihara monoids, they are also isomorphic as i-lattices. Lemma 6.2.18 gives that $(\mathbf{A}_{\bowtie})^{\bowtie}$ and $\mathcal{E}(\mathcal{S}(\mathbf{A}_{\bowtie}), \subseteq \cup \supseteq)$ are isomorphic as i-lattices, and hence $\mathbf{A}$ is isomorphic as an i-lattice to both

$\mathcal{E}(\mathcal{S}(\mathbf{A}), \subseteq \cup \supseteq)$ and $\mathcal{E}\mathcal{D}(\mathbf{A})$. This implies that

$$(\mathcal{S}(\mathbf{A}_{\bowtie}), \subseteq \cup \supseteq) \cong \mathcal{D}\mathcal{E}(\mathcal{S}(\mathbf{A}), \subseteq \cup \supseteq) \cong \mathcal{D}\mathcal{E}\mathcal{D}(\mathbf{A}) \cong \mathcal{D}(\mathbf{A})$$

as pointed Kleene spaces. Pick a pKS-isomorphism $\alpha\colon \mathcal{D}(\mathbf{A}) \to (\mathcal{S}(\mathbf{A}_{\bowtie}), \subseteq \cup \supseteq)$. Note that for $h, k \in \mathcal{D}(\mathbf{A})$,

$$\begin{aligned} h\ Q_{\mathbf{A}}\ k &\iff \alpha(h) \text{ and } \alpha(k) \text{ are } \subseteq\text{-comparable} \\ &\iff \alpha(h) \subseteq \alpha(k) \text{ or } \alpha(k) \subseteq \alpha(h) \\ &\iff h \lesssim k \text{ or } k \lesssim h \\ &\iff h \text{ and } k \text{ are } \lesssim\text{-comparable.} \end{aligned}$$

Hence $Q_{\mathbf{A}}$ is the relation of $\lesssim$-comparability, which proves the claim. $\square$

Lemma 6.2.23. *Let $\mathbf{X} = (X, \leqslant, D, \leqslant \cup \geqslant, \top, \tau)$ be a Sugihara space. Then $\mathcal{E}(\mathbf{X})$ is a Sugihara monoid.*

Proof. Note that $(X, \leqslant, D, \top, \tau)$ is bRS-space by Lemma 6.2.4. It follows that $\mathcal{E}(X, \leqslant, D, \top, \tau) \in \mathsf{bRSA}$. By Lemma 6.2.18 we get that $\mathcal{E}(\mathcal{S}\mathcal{A}(X, \leqslant, D, \top, \tau), \subseteq \cup \supseteq)$ is isomorphic as an i-lattice to $\mathcal{A}(X, \leqslant, D, \top, \tau)^{\bowtie}$. We have also that

$$\mathcal{S}\mathcal{A}(X, \leqslant, D, \top, \tau) \cong (X, \leqslant, D, \top, \tau)$$

as bRS-spaces, whence

$$\mathcal{A}(X, \leqslant, D, \top, \tau)^{\bowtie} \cong \mathcal{E}((X, \leqslant, D, \top, \tau), \leqslant \cup \geqslant)$$

as i-lattices. The last of these is exactly the i-lattice reduct of $\mathcal{E}(\mathbf{X})$, so it follows that $\mathcal{E}(\mathbf{X})$ is isomorphic as an i-lattice to the Sugihara monoid $\mathcal{A}(X, \leqslant, D, \top, \tau)^{\bowtie}$.

The operations $\to$ and $\cdot$ hence make the i-lattice reduct of $\mathcal{E}(\mathbf{X})$ into a Sugihara monoid by transport of structure. $\square$

Lemma 6.2.24. *Let* $\mathbf{A}, \mathbf{B} \in \mathsf{SM}$ *and let* $h\colon \mathbf{A} \to \mathbf{B}$ *be a morphism in* SM*. Then* $\mathcal{D}(h) = \xi_{\mathbf{A}}^{-1} \circ \mathcal{D}(h_{\bowtie}) \circ \xi_{\mathbf{B}}$.

Proof. Let $x \in \mathcal{D}(B)$ and let $a \in A$. Note that if $a \in A^{-}$, then

$$h_{\bowtie}(a) = h{\restriction}_{A^{-}}(a) = h(a).$$

Moreover, $h{\restriction}_{A^{-}}^{-1}[B^{-}] = A^{-}$. From these observations, we have

$$\begin{aligned}
a \in (\xi_{\mathbf{A}} \circ \mathcal{D}(h))(x) &\iff a \in \xi_{\mathbf{A}}(x \circ h) \\
&\iff a \in (x \circ h)^{-1}[\{0,1\}] \cap A^{-} \\
&\iff (x \circ h)(a) \in \{0,1\} \text{ and } a \in A^{-} \\
&\iff (x \circ h_{\bowtie})(a) \in \{0,1\} \text{ and } a \in A^{-} \\
&\iff x(h{\restriction}_{A^{-}}(a)) \in \{0,1\} \text{ and } a \in A^{-} \\
&\iff a \in h{\restriction}_{A^{-}}^{-1}[x^{-1}[\{0,1\}]] \cap A^{-} \\
&\iff a \in h{\restriction}_{A^{-}}^{-1}[x^{-1}[\{0,1\}] \cap B^{-}] \\
&\iff a \in \mathcal{S}(h_{\bowtie})(x^{-1}[\{0,1\}] \cap B^{-}) \\
&\iff a \in \mathcal{S}(h_{\bowtie})(\xi_{\mathbf{B}}(x)) \\
&\iff a \in (\mathcal{S}(h_{\bowtie}) \circ \xi_{\mathbf{B}})(x).
\end{aligned}$$

Hence $\xi_{\mathbf{A}} \circ \mathcal{D}(h) = \mathcal{S}(h_{\bowtie}) \circ \xi_{\mathbf{B}}$. As $\xi_{\mathbf{A}}$ is an isomorphism of bRS-spaces by Lemma 6.2.16, it has an inverse $\xi_{\mathbf{A}}^{-1}$. This implies that $\mathcal{D}(h) = \xi_{\mathbf{A}}^{-1} \circ \mathcal{S}(h_{\bowtie}) \circ \xi_{\mathbf{B}}$. $\square$

Corollary 6.2.25. *Let* $\mathbf{A}, \mathbf{B} \in \mathsf{SM}$ *and let* $h\colon \mathbf{A} \to \mathbf{B}$ *be a morphism in* SM*. Then* $\mathcal{D}(h)$ *is a morphism of* pSS*.*

Proof. Lemma 6.2.24 writes $\mathcal{D}(h)$ as a composition of bRSS-morphisms, and hence $\mathcal{D}(h)$ is a bRSS-morphism. $\square$

Lemma 6.2.26. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be Sugihara spaces and let* $\alpha\colon \mathbf{X} \to \mathbf{Y}$ *be a morphism in* pSS. *Then* $\mathcal{E}(\alpha) = \mu_{\mathbf{X}} \circ \mathcal{A}(\alpha)^{\bowtie} \circ \mu_{\mathbf{Y}}^{-1}$.

Proof. Let $(U, V) \in \mathcal{A}(Y)^{\bowtie}$ and let $x \in X$. Then

$$\begin{aligned}((\mu_{\mathbf{X}} \circ \mathcal{A}(\alpha)^{\bowtie})(U,V))(x) &= \mu_{\mathbf{X}}(\mathcal{A}(\alpha)^{\bowtie}(U,V))(x)\\ &= \mu_{\mathbf{X}}(\mathcal{A}(\alpha)(U), \mathcal{A}(\alpha)(V))(x)\\ &= \mu_{\mathbf{X}}(\alpha^{-1}[U], \alpha^{-1}[V])(x)\\ &= C_{\alpha^{-1}[U],\alpha^{-1}[V]}(x).\end{aligned}$$

Also note,

$$\begin{aligned}((\mathcal{E}(\alpha) \circ \mu_{\mathbf{Y}})(U,V))(x) &= \mathcal{E}(\alpha)(\mu_{\mathbf{Y}}(U,V))(x)\\ &= (C_{U,V} \circ \alpha)(x)\\ &= C_{U,V}(\alpha(x)).\end{aligned}$$

Observe that $\alpha(x) \in U$ if and only if $x \in \alpha^{-1}[U]$, $\alpha(x) \in V$ if and only if $x \in \alpha^{-1}[V]$, and $\alpha(x) \in U \cap V$ if and only if $x \in \alpha^{-1}[U \cap V] = \alpha^{-1}[U] \cap \alpha^{-1}[V]$. From the definition of $C_{U,V}$ we get

$$C_{U,V}(\alpha(x)) = C_{\alpha^{-1}[U],\alpha^{-1}[V]}(x).$$

This yields $\mu_{\mathbf{X}} \circ \mathcal{A}(\alpha)^{\bowtie} = \mathcal{E}(\alpha) \circ \mu_{\mathbf{Y}}$. As $\mu_{\mathbf{Y}}$ is a Sugihara monoid isomorphism (thus invertible), it follows that $\mathcal{E}(\alpha) = \mu_{\mathbf{X}} \circ \mathcal{A}(\alpha)^{\bowtie} \circ \mu_{\mathbf{Y}}^{-1}$. $\square$

Corollary 6.2.27. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be Sugihara spaces and let* $\alpha\colon \mathbf{A} \to \mathbf{B}$ *be a morphism in* pSS*. Then* $\mathcal{E}(\alpha)$ *is a morphism of* SM*.*

Proof. $\mathcal{E}(\alpha)$ is the composition of morphisms in SM by Lemma 6.2.26, so $\mathcal{E}(\alpha)$ is a morphism of SM. □

Lemma 6.2.28. *Let* $\mathbf{A} \in \mathsf{SM}$*. Then* $\mathcal{ED}(\mathbf{A}) \cong \mathbf{A}$*.*

Proof. $\mathcal{D}(\mathbf{A})$ and $\mathcal{S}(\mathbf{A}_{\bowtie})$ are isomorphic as bRS-spaces to via $\xi_{\mathbf{A}}$. Moreover, we have $\mathcal{AS}(\mathbf{A}_{\bowtie}) \cong \mathbf{A}_{\bowtie}$ as bRS-algebras, and thus $(\mathcal{AS}(\mathbf{A}_{\bowtie}))^{\bowtie} \cong (\mathbf{A}_{\bowtie})^{\bowtie} \cong \mathbf{A}$ as Sugihara monoids. Since $\mu_{\mathcal{S}(\mathbf{A}_{\bowtie})}$ is a Sugihara monoid isomorphism from $(\mathcal{AS}(\mathbf{A}_{\bowtie}))^{\bowtie}$ to $\mathcal{E}(\mathcal{S}(\mathbf{A}_{\bowtie}), \subseteq \cup \supseteq)$ from Remark 6.2.21, we obtain $\mathcal{ED}(\mathbf{A}) \cong \mathbf{A}$ as claimed. □

Lemma 6.2.29. *Let* $\mathbf{X} = (X, \leqslant, \leqslant \cup \geqslant, D, \top, \tau)$ *be a Sugihara space. Then* $\mathcal{DE}(\mathbf{X}) \cong \mathbf{X}$*.*

Proof. Note that $\mathcal{E}(\mathbf{X})$ is isomorphic as a Sugihara monoid to $\mathcal{A}(X, \leqslant, D, \top, \tau)^{\bowtie}$ via $\mu_{\mathbf{X}}$. Also, $\mathcal{DE}(\mathbf{X})$ and $\mathcal{S}(\mathcal{E}(\mathbf{X})_{\bowtie})$ are isomorphic as bRS-spaces via $\xi_{\mathcal{E}(\mathbf{X})}$. Thus $\mathcal{DE}(\mathbf{X})$ and $\mathcal{S}((\mathcal{A}(X, \leqslant, D, \top, \tau)^{\bowtie})_{\bowtie})$ are isomorphic as bRS-spaces. The last of these is isomorphic to $(X, \leqslant, D, \top, \tau)$, whence $\mathcal{DE}(\mathbf{X})$ and $(X, \leqslant, D, \top, \tau)$ are isomorphic as bRS-spaces. The bRSS-isomorphism witnessing this is a pSS-isomorphism between $\mathcal{DE}(\mathbf{X})$ and $(X, \leqslant, \leqslant \cup \geqslant, D, \top, \tau)$ by definition, and the latter is exactly $\mathbf{X}$. □

Theorem 6.2.30. SM *and* pSS *are dually-equivalent.*

Proof. This follows from Lemmas 6.2.22, 6.2.23, 6.2.24, 6.2.26, 6.2.28, 6.2.29, and Corollaries 6.2.25 and 6.2.27. Functoriality and naturality are immediate from the Davey-Werner duality. □

Of course, *mutatis mutandis* all of the above applies to bounded Sugihara monoids as well.

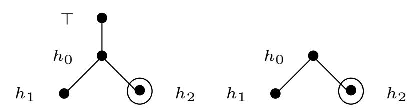


Figure 6.2: Hasse diagrams for $\mathcal{D}(\mathbf{E})$ and $\mathcal{D}(\mathbf{E}_\perp)$

Definition 6.2.31. *A Kleene space* $(X, \leqslant, Q, D, \tau)$ *is called an* unpointed Sugihara space *if*

1. $(X, \leqslant, \tau)$ *is an Esakia space,*

2. Q *is the relation of comparability with respect to* $\leqslant$*, i.e.,* $Q = \leqslant \cup \geqslant$*, and*

3. D *is open.*

We typically say that $(X, \leqslant, D, \tau)$ *is an unpointed Sugihara space, leaving* Q *to be inferred.*

A bGS*-morphism between unpointed Sugihara spaces is called an* unpointed Sugihara space morphism*, and we denote the category of unpointed Sugihara spaces with unpointed Sugihara space morphisms by* SS*.*

The arguments above apply to the bounded setting with only trivial modification, and we may obtain the following.

Corollary 6.2.32. $\mathsf{SM}_\perp$ *and* SS *are dually-equivalent.*

Example 6.2.33. *Recall the Sugihara monoid* $\mathbf{E}$ *of Example 2.3.11. Figure 6.2 gives the labeled Hasse diagram of* $\mathcal{D}(\mathbf{E})$*, where the maps* $\top, h_0, h_1, h_2$ *are uniquely determined by* $\top(a) = 0$ *for all* $a \in E$*,* $h_0(a) = 0$ *for all* a *except* $(2,2), (-2,-2)$*,* $h_1(a) = 0$ *for* $a = (0,1)$ *or* $(0,-1)$*, and* $h_2(a) = 1$ *for all* $a \in {\uparrow}(-1,1)$ *and* $h_2(a) = -1$ *for* $a \in {\downarrow}(1,-1)$*. Observe that of these maps, only* h_2 *lies in the designated subset (i.e., since its image does not contain* 0*). Letting* $\mathbf{E}_\perp$ *be the expansion of* $\mathbf{E}$ *by* *universal lattice bounds, we may obtain the dual of* $\mathbf{E}_\perp$ *by excluding the map* $\top$ *(i.e., since* $\top$ *is not a morphism in the bounded signature).*

6.3 Another formulation of the duality

Topological dualities of different kinds offer different strengths. In contrast to Esakia duality, the topological side of a natural duality is well-behaved on a categorical level (e.g., products may be computed as Cartesian products). However, natural dualities lack much of the pictorial insight that drives Priestley duality and its various modifications. As a restriction of the Davey-Werner natural duality, the duality for Sugihara monoids articulated in this chapter is less geometric in character than Priestley duality. This final section of Chapter 6 aims to offer some pictorial insight.

If $\mathbf{A}$ is an *odd* Sugihara monoid, we may understand its dual in terms of certain algebraic substructures that are ordered by containment. This representation by *convex prime subalgebras* has much of the pictorial flavor of the Esakia duality's representation of duals by prime filers.

For Sugihara monoids that are not odd, the convex prime subalgebra representation is unavailable. In its stead we offer another representation in terms of certain filters, a perspective that proves important in Chapter 7.

Definition 6.3.1. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e, \neg)$ *be an odd Sugihara monoid. A* $(\wedge, \vee, e, \neg)$*-subalgebra* $\mathbf{C}$ *of* $\mathbf{A}$ *is said to be a* convex prime subalgebra *if for all* $a, b, c \in A$,

1. *If* $a, c \in C$ *and* $a \leqslant b \leqslant c$, *then* $b \in C$, *and*

2. *If* $a \wedge b \in C$, *then* $a \in C$ *or* $b \in C$.

We designate the collection of convex prime subalgebras of $\mathbf{A}$ *by* $\mathrm{Conv}(\mathbf{A})$.

If $\mathbf{C} \in \mathrm{Conv}(\mathbf{A})$ and $a \vee b \in C$, then $\neg a \wedge \neg b = \neg(a \vee b) \in C$. Hence $\neg a \in C$ or $\neg b \in C$, so $a \in C$ or $b \in C$ by $\neg$-closure. It follows that each convex prime subalgebra is prime with respect to $\vee$ as well as $\wedge$.

Proposition 6.3.2. *Let* $\mathbf{A} \in \mathsf{OSM}$*. Then* $\mathcal{D}(\mathbf{A})$ *is order isomorphic to the poset* $(\mathrm{Conv}(\mathbf{A}), \subseteq)$.

Proof. Note that $\mathcal{D}(\mathbf{A})$ is order isomorphic to $\mathcal{S}(\mathbf{A}_{\bowtie})$ from Lemma 6.2.9. Thus it is enough to show $(\mathrm{Conv}(\mathbf{A}), \subseteq)$ is order isomorphic to $\mathcal{S}(\mathbf{A}_{\bowtie})$. Define a function $\Omega\colon \mathrm{Conv}(\mathbf{A}) \to \mathcal{S}(\mathbf{A}_{\bowtie})$ by $\Omega(\mathbf{C}) = C \cap A^{-}$.

We first show that $\Omega(\mathbf{C})$ is a filter. If $a \in \Omega(\mathbf{C})$ and $b \in A^{-}$ with $a \leqslant b$, then $a \leqslant b \leqslant e \in C$ implies $b \in C$ by convexity. Thus $\Omega(\mathbf{C})$ is an up-set. For closure under meets, let $a, b \in \Omega(\mathbf{C})$. Then $\mathbf{C}$ being $\wedge$-closed implies $a \wedge b \in C$, and $a, b \leqslant e$ implies $a \wedge b \leqslant e$. Hence $a \wedge b \in \Omega(\mathbf{C})$, so $\Omega(\mathbf{C})$ is a filter.

For primality, let $a, b \in A^{-}$ with $a \vee b \in \Omega(\mathbf{C})$. Then $a \vee b \in C$ and $a \vee b \leqslant e$. The latter gives $a \leqslant e$ and $b \leqslant e$, so one of $a \in \Omega(\mathbf{C})$ or $b \in \Omega(\mathbf{C})$ follows from the $\vee$-primality of $\mathbf{C}$. Hence Ω is well-defined.

Ω is obviously isotone. To prove that Ω reflects the order, let $\mathbf{C}_1, \mathbf{C}_2 \in \mathrm{Conv}(\mathbf{A})$ such that $\Omega(\mathbf{C}_1) \subseteq \Omega(\mathbf{C}_2)$ and let $a \in C_1$. Then $\neg a \in C_1$, and moreover we have that $a \wedge e, \neg a \wedge e \in \Omega(\mathbf{C}_1)$, whence $a \wedge e, \neg a \wedge e \in \Omega(\mathbf{C}_2)$. Since $a \wedge e, \neg a \wedge e \in \Omega(\mathbf{C}_2)$, it follows that $a \wedge e, \neg a \wedge e \in C_2$. From the fact that $\neg a \wedge e \in C_2$, we obtain that $\neg(\neg a \wedge e) = a \vee \neg e \in C_2$. As $a \wedge e \leqslant a \leqslant a \vee \neg e$, convexity gives $a \in C_2$. Therefore $C_1 \subseteq C_2$.

We now show Ω is onto, so let $\mathfrak{x} \in \mathcal{S}(A_{\bowtie})$. Let

$$\uparrow_{\mathbf{A}}\mathfrak{x} = \{a \in A : (\exists p \in \mathfrak{x})(p \leqslant a)\},$$

$$\neg\mathfrak{x} = \{\neg a : a \in \mathfrak{x}\},$$

$$\downarrow_{\mathbf{A}}\neg\mathfrak{x} = \{a \in A : (\exists p \in \neg\mathfrak{x})(a \leqslant p)\}, \text{ and}$$

$$C = \uparrow_{\mathbf{A}}\mathfrak{x} \cap \downarrow_{\mathbf{A}}\neg\mathfrak{x}.$$

We claim that C is the universe of a convex prime subalgebra $\mathbf{C}$, and that $\Omega(\mathbf{C}) = \mathfrak{x}$.

First, note that since $\mathfrak{x} \in \mathcal{S}(A_{\bowtie})$ we have that $e \in \mathfrak{x}$, whence $e \in C$.

Second, observe that if $a \in C$, then there exists $p, q \in \mathfrak{x}$ such that $p \leqslant a \leqslant \neg q$. Then $q \leqslant \neg a \leqslant \neg p$, so $\neg a \in C$.

Third, suppose that $a, b \in C$. Then there are $p_1, p_2, q_1, q_2 \in \mathfrak{x}$ so that $p_1 \leqslant a \leqslant \neg q_1$ and $p_2 \leqslant b \leqslant \neg q_2$. This yields

$$p_1 \wedge p_2 \leqslant a \wedge b \leqslant \neg q_1 \wedge \neg q_2 = \neg(q_1 \vee q_2).$$

Since $\mathfrak{x}$ is a filter, $p_1 \wedge p_2, q_1 \vee q_2 \in \mathfrak{x}$. Thus $a \wedge b \in C$. Moreover, since

$$p_1 \vee p_2 \leqslant a \vee b \leqslant \neg q_1 \vee \neg q_2 = \neg(q_1 \wedge q_2)$$

we have $a \vee b \in C$. Since $e \in \mathfrak{x}$, $e \leqslant e \leqslant \neg e = e$ gives $e \in C$, and this shows that C is a $(\wedge, \vee, \neg, e)$-subalgebra.

To see that C is convex, suppose that $a, c \in C$ and $b \in A$ with $a \leqslant b \leqslant c$. Since $a, c \in C$, there are $p_1, p_2, q_1, q_2 \in \mathfrak{x}$ with $p_1 \leqslant a \leqslant \neg q_1$ and $p_2 \leqslant c \leqslant \neg q_2$. This gives $p_1 \leqslant a \leqslant b \leqslant c \leqslant \neg q_2$, so $b \in C$ as well. Thus $\mathbf{C}$ is a convex prime subalgebra.

Finally, to prove $\Omega(\mathbf{C}) = \mathfrak{x}$, suppose that $a \in \Omega(\mathbf{C}) = C \cap A^-$. Then there exists $p, q \in x$ with $p \leqslant a \leqslant \neg q$, and $a \in A^-$. Since $\mathfrak{x}$ is and up-set, $p \leqslant a$ and $p \in \mathfrak{x}$ implies $a \in \mathfrak{x}$. Hence $\Omega(\mathbf{C}) \subseteq \mathfrak{x}$. On the other hand, if $a \in \mathfrak{x}$, then $a \leqslant a \leqslant e = \neg e$ gives that $a \in \Omega(\mathbf{C})$ as desired. This proves the proposition. $\square$

If $\mathbf{A}$ is a Sugihara monoid (or bounded Sugihara monoid) with monoid identity e, define[17]

$$I(\mathbf{A}) := \{\mathfrak{x} \in \mathcal{S}(A) : e \in \mathfrak{x}\}.$$

The set $I(\mathbf{A})$ provides us with a pictorial representation of the dual of an arbitrary Sugihara monoid.

Proposition 6.3.3. *Let* $\mathbf{A} \in \mathsf{SM} \cup \mathsf{SM}_\perp$*. Then* $\mathcal{E}(\mathbf{A})$ *is order isomorphic to the poset* $(I(\mathbf{A}), \subseteq)$*.*

Proof. Define $\Omega_\mathbf{A} \colon \mathcal{E}(\mathbf{A}) \to I(\mathbf{A})$ by $\Omega(h) = h^{-1}[\{0,1\}]$. That $\{0,1\}$ is a prime filter and h is a $(\wedge, \vee)$-homomorphism implies $\Omega_\mathbf{A}(h) \in \mathcal{S}(A)$. Also, $h(e) \in \{0,1\}$ implies $e \in h^{-1}[\{0,1\}]$ for each $h \in \mathcal{D}(A)$, whence $\Omega_\mathbf{A}$ is well-defined.

An identical proof to that offered in Lemma 6.2.6 shows $\Omega_\mathbf{A}$ is order-preserving. To prove $\Omega_\mathbf{A}$ is order-reflecting, let $h_1, h_2 \in \mathcal{D}(A)$ with $\Omega_\mathbf{A}(h_1) \subseteq \Omega_\mathbf{A}(h_2)$. Were it the case that $h_1 \not\lesssim h_2$, then there exists $a \in A$ such that $h_2(a) = -1$ and $h_1(a) \neq -1$, or else $h_2(a) = 1$ and $h_1(a) \neq 1$.

For the first case, we have that $h_1(a) \in \{0,1\}$. Then $a \in \Omega_\mathbf{A}(h_1) \subseteq \Omega_\mathbf{A}(h_2)$, so $h_2(a) \in \{0,1\}$, a contradiction. For the second case, $h_1(a) \in \{-1,0\}$, and it follows that $h_1(\neg a) \in \{0,1\}$. Then $h_2(\neg a) \in \{0,1\}$, but this contradicts $h_2(a) = 1$. Therefore $h_1 \lesssim h_2$.

Finally, to see that $\Omega_\mathbf{A}$ is onto, let $\mathfrak{x} \in I(\mathbf{A})$ and set $\neg\mathfrak{x} = \{\neg a : a \in \mathfrak{x}\}$. From $e \in \mathfrak{x}$ and the identity $e \leqslant a \vee \neg a$ we get $a \vee \neg a \in \mathfrak{x}$ for all $a \in A$, whence by primality $a \in \mathfrak{x}$ or $\neg a \in \mathfrak{x}$. This implies $a \in \mathfrak{x}$ or $a \in \neg\mathfrak{x}$, and therefore each $a \in A$ is contained in exactly one of the disjoint sets $\mathfrak{x} - \neg\mathfrak{x}$, $\mathfrak{x} \cap \neg\mathfrak{x}$, or $\neg\mathfrak{x} - \mathfrak{x}$. We may define a map

[17] Observe that $I(\mathbf{A})$ is the subset encoding the monoid identity in the extended Priestley duality (see Section 3.4).

$h\colon A \to \{-1, 0, 1\}$ by

$$h(a) = \begin{cases} 1 & a \in \mathfrak{x} - \neg\mathfrak{x} \\ 0 & a \in \mathfrak{x} \cap \neg\mathfrak{x} \\ -1 & a \in \neg\mathfrak{x} - \mathfrak{x} \end{cases}$$

Case analysis readily shows that h is a homomorphism with respect to $\wedge, \vee, \neg$, and the lattice bounds (when applicable). Hence $h \in \mathcal{E}(A)$. Also,

$$\Omega_{\mathbf{A}}(h) = h^{-1}[\{0, 1\}] = h^{-1}[\{0\}] \cup h^{-1}[\{1\}] = (\mathfrak{x} - \neg\mathfrak{x}) \cup (\mathfrak{x} \cap \neg\mathfrak{x}) = \mathfrak{x}.$$

This provides $\Omega_{\mathbf{A}}$ is a surjection. Because $\Omega_{\mathbf{A}}$ is a order-preserving, order-reflecting, and onto, the result follows. $\square$

Chapter 7

Dualized representations of Sugihara monoids

Previous chapters have articulated two distinct topological dualities for bounded Sugihara monoids:

- The extended Priestley duality linking $\mathsf{SM}_{\perp}$ and $\mathsf{SM}_{\perp}^{\tau}$ (Section 3.4), which is a functional duality in the sense of Chapter 4.
- The Esakia-style duality linking $\mathsf{SM}_{\perp}$ and SS (Chapter 6).

These two dualities have a rather different character. The extended Priestley duality achieves categorical equivalence by expanding the structure of duals of a suitably-chosen reduct. In contrast, the Esakia-style duality achieves equivalence by identifying a reduct that completely determines algebras in the full signature, and then pinpointing the duals of algebras that arise as such reducts.

In addition to the above, $\mathsf{SM}_\perp$ also enjoys a covariant equivalence to bGA via the functors $(-)_\bowtie$ and $(-)^\bowtie$ (see Section 5.3). However, the construction of a bounded Sugihara monoid from a bG-algebra is a rather complicated affair, as the definitions of the operations inherent in the functor $(-)^\bowtie$ attest. The chief goal of this chapter is to provide a dualized account of the covariant equivalence given by $(-)_\bowtie$ and $(-)^\bowtie$, in particular offering a greatly simplified presentation of the construction underlying $(-)^\bowtie$ on duals. This project implicates both of the dualities for $\mathsf{SM}_\perp$, and in fact the connection between the two dualities is the key to understanding $(-)_\bowtie$ and $(-)^\bowtie$ in duality-theoretic terms. The results of this chapter come from the author's [24].

We proceed as follows. First, in Section 7.1 we provide a dual analogue of the functor $(-)_\bowtie$ that constructs an object of SS from an object of $\mathsf{SM}_\perp^\tau$. Then in Section 7.2 we present a construction of objects of $\mathsf{SM}_\perp^\tau$ from objects of SS, yielding a dual analogue of $(-)^\bowtie$. Lastly, in Section 7.3 we tie these two constructions together and attend to categorical details.

7.1 Dual enriched negative cones

In order to present a dual analogue of the functor $(-)_\bowtie$, we first require some technical results. Given $\mathbf{A} \in \mathsf{SM}_\perp$, recall that $I(\mathbf{A}) = \{\mathfrak{x} \in \mathcal{S}(A) : e \in \mathfrak{x}\}$, and that $\Omega_\mathbf{A} \colon \mathcal{D}(\mathbf{A}) \to I(\mathbf{A})$ defined by

$$\Omega_\mathbf{A}(h) = h^{-1}[\{0, 1\}]$$

is an order isomorphism between $\mathcal{D}(\mathbf{A})$ and $(I(\mathbf{A}), \subseteq)$ (see Proposition 6.3.3).

Lemma 7.1.1. *When $I(\mathbf{A})$ is given with the topology inherited as a subspace of $\mathcal{S}(\mathbf{A})$, $\Omega_\mathbf{A}$ is continuous.*

Proof. We show that the inverse image of each subbasis element is open. The subbasis elements of $I(\mathbf{A})$ have the form

$$\varphi(a) = \{\mathfrak{x} \in I(\mathbf{A}) : a \in \mathfrak{x}\}$$

$$\varphi(a)^{\mathsf{c}} = \{\mathfrak{x} \in I(\mathbf{A}) : a \notin \mathfrak{x}\}.$$

With this in mind, we have for each $a \in A$ that

$$\begin{aligned}
\Omega_{\mathbf{A}}^{-1}[\varphi(a)] &= \{h \in \mathcal{D}(A) : \Omega_{\mathbf{A}}(h) \in \varphi(a)\} \\
&= \{h \in \mathcal{D}(A) : a \in h^{-1}[\{0,1\}]\} \\
&= \{h \in \mathcal{D}(A) : h(a) \in \{0,1\}\} \\
&= \{h \in \mathcal{D}(A) : h(a) = 0\} \cup \{h \in \mathcal{D}(A) : h(a) = 1\}
\end{aligned}$$

Each of the latter sets is a subbasis element of $\mathcal{E}(\mathbf{A})$ by Lemma 3.3.10. Moreover,

$$\begin{aligned}
\Omega_{\mathbf{A}}^{-1}[\varphi(a)^{\mathsf{c}}] &= \{h \in \mathcal{D}(A) : \Omega_{\mathbf{A}}(h) \in \varphi(a)^{\mathsf{c}}\} \\
&= \{h \in \mathcal{D}(A) : a \notin h^{-1}[\{0,1\}]\} \\
&= \{h \in \mathcal{D}(A) : h(a) \notin \{0,1\}\} \\
&= \{h \in \mathcal{D}(A) : h(a) = -1\}
\end{aligned}$$

The above is also a subbasis element, which proves the claim. $\square$

Lemma 7.1.2. $\Omega_{\mathbf{A}}$ *is a homeomorphism.*

Proof. Note that $\mathcal{S}(\mathbf{A})$ is a Hausdorff space, whence its subspace $I(\mathbf{A})$ is also Hausdorff. $\mathcal{D}(\mathbf{A})$ is compact because it is a Priestley space. Hence $\Omega_{\mathbf{A}}$ is a continuous bijection from a compact space to a Hausdorff space, and therefore a homeomorphism. $\square$

From the foregoing observations, we get the following.

Lemma 7.1.3. *$I(\mathbf{A})$ is an Esakia space.*

Proof. We first show that $I(\mathbf{A})$ is a Priestley space. Note that $I(\mathbf{A})$ is compact since $\Omega_{\mathbf{A}}$ is a homeomorphism of $I(\mathbf{A})$ with a compact space. Let $\mathfrak{x}, \mathfrak{y} \in I(\mathbf{A})$ such that $\mathfrak{x} \nsubseteq \mathfrak{y}$. This implies that $\Omega_{\mathbf{A}}^{-1}(\mathfrak{x}) \nleq \Omega_{\mathbf{A}}^{-1}(\mathfrak{y})$ since $\Omega_{\mathbf{A}}^{-1}$ is an order isomorphism. Because $\mathcal{D}(\mathbf{A})$ is a Priestley space among other things, there exists a clopen up-set $U \subseteq \mathcal{D}(\mathbf{A})$ such that $\Omega_{\mathbf{A}}^{-1}(\mathfrak{x}) \in U$ and $\Omega_{\mathbf{A}}^{-1}(\mathfrak{y}) \notin U$. This implies $\Omega_{\mathbf{A}}[U]$ is a clopen up-set of $I(\mathbf{A})$ and $\mathfrak{x} \in \Omega_{\mathbf{A}}[U]$ and $\mathfrak{y} \notin \Omega_{\mathbf{A}}[U]$, showing that $I(\mathbf{A})$ is a Priestley space.

For the rest, note that $\Omega_{\mathbf{A}}$ is an order isomorphism and a homeomorphism. This means that $\Omega_{\mathbf{A}}$ is an isomorphism of Priestley spaces. As $I(\mathbf{A})$ is a Priestley space that is isomorphic to the Esakia space $\mathcal{D}(\mathbf{A})$, we have that $I(\mathbf{A})$ is an Esakia space too. $\square$

Recall that for a prime filter $\mathfrak{x}$ of $\mathbf{A}$, we have

$$\mathfrak{x}^* = \{a \in A : \neg a \notin \mathfrak{x}\}.$$

Note that if $\mathbf{A}$ is involutive (and in particular a bounded Sugihara monoid), then $\mathfrak{x}^{**} = \mathfrak{x}$.

Lemma 7.1.4. *Let $\mathbf{A} \in \mathsf{SM}_\bot$. The following hold for all $\mathfrak{x} \in \mathcal{S}(A)$.*

1. *$\mathfrak{x} \in I(\mathbf{A})$ or $\mathfrak{x}^* \in I(\mathbf{A})$.*

2. *$\mathfrak{x} \subseteq \mathfrak{x}^*$ or $\mathfrak{x}^* \subseteq \mathfrak{x}$.*

3. *The larger of $\mathfrak{x}$ and $\mathfrak{x}^*$ lies in $I(\mathbf{A})$.*

4. *The following are equivalent.*

(a) $\mathfrak{x} = \mathfrak{x}^*$,

(b) $e \in \mathfrak{x}$ *and* $\neg e \notin \mathfrak{x}$,

(c) $\mathfrak{x}, \mathfrak{x}^* \in I(\mathbf{A})$.

Proof. For (1), suppose $e \notin \mathfrak{x}$. Since $\neg e \leqslant e$, we have $\neg e \notin \mathfrak{x}$ too. Hence $e \in \mathfrak{x}^*$.

For (2), assume that $\mathfrak{x} \nsubseteq \mathfrak{x}^*$. Then there exists $a \in \mathfrak{x}$ with $a \notin \mathfrak{x}^*$. The latter provides that $\neg a \in \mathfrak{x}$, whence we obtain $a \wedge \neg a \in \mathfrak{x}$. Let $b \in \mathfrak{x}^*$. Then $\neg b \notin \mathfrak{x}$. By the normality of the i-lattice reduct of $\mathbf{A}$ we get that $a \wedge \neg a \leqslant b \vee \neg b$, so $b \vee \neg b \in \mathfrak{x}$. Since $\mathfrak{x}$ is prime, this implies that $b \in \mathfrak{x}$ or $\neg b \in \mathfrak{x}$. But the latter is a contradiction, so we get $b \in \mathfrak{x}$ and hence $\mathfrak{x}^* \subseteq \mathfrak{x}$.

from (1) we may suppose without loss of generality that $e \in \mathfrak{x}^*$. Let $a \in \mathfrak{x}$. If $a \notin \mathfrak{x}^*$, then $\neg(\neg a) \notin \mathfrak{x}$, whence $\neg a \in \mathfrak{x}$. It follows that $a, \neg a \in \mathfrak{x}$, so $a \wedge \neg a \leqslant \neg e$ gives $\neg e \in \mathfrak{x}$. This is a contradiction, so $\mathfrak{x} \subseteq \mathfrak{x}^*$ follows.

(3) is obvious from (1) and (2).

For (4), we prove first (a) implies (b), so suppose $\mathfrak{x} = \mathfrak{x}^*$. If $e \notin \mathfrak{x}$, then $e = \neg\neg e \notin \mathfrak{x}$, whence $\neg e \in \mathfrak{x}^*$. It follows that $\neg e \in \mathfrak{x}$. But $\neg e \leqslant e$ implies that $e \in \mathfrak{x}$, so this is impossible. Thus $e \in \mathfrak{x}$, and $e \in \mathfrak{x}^*$ as well. Were $\neg e \in \mathfrak{x}$, we would have $\neg e \in \mathfrak{x}^*$ and this implies $\neg\neg e \notin \mathfrak{x}$. This is a contradiction to $e \in \mathfrak{x}$, whence $e \in \mathfrak{x}$ and $\neg e \notin \mathfrak{x}$.

For (b) implies (c), suppose that $e \in \mathfrak{x}$ and $\neg e \notin \mathfrak{x}$. The second of these provides that $e \in \mathfrak{x}^*$, whence $\mathfrak{x}, \mathfrak{x}^* \in I(\mathbf{A})$ is immediate.

For (c) implies (a), suppose $\mathfrak{x}, \mathfrak{x}^* \in I(\mathbf{A})$. This means $e \in \mathfrak{x}, \mathfrak{x}^*$, so $e \in \mathfrak{x}$ and $\neg e \notin \mathfrak{x}$. Let $a \in \mathfrak{x}$. Were it the case that $\neg a \in \mathfrak{x}$, we would have $a, \neg a \in \mathfrak{x}$, which implies $a \wedge \neg a \leqslant \neg e \in \mathfrak{x}$, a contradiction. This gives $\neg a \notin \mathfrak{x}$, whence $a \in \mathfrak{x}^*$ and $\mathfrak{x} \subseteq \mathfrak{x}^*$. For the other inclusion, let $a \in \mathfrak{x}^*$. Then $\neg a \notin \mathfrak{x}$. Note that $a \vee \neg a \geqslant e$ and $e \in \mathfrak{x}$ gives $a \vee \neg a \in \mathfrak{x}$, whence $a \in \mathfrak{x}$ by primality. Therefore $\mathfrak{x}^* \subseteq \mathfrak{x}$, and we get equality. This settles (4). □

For $\mathbf{A} \in \mathsf{SM}_\perp$, let $\mathcal{S}(\mathbf{A}) = (\mathcal{S}(A), \subseteq, R, ^*, I(\mathbf{A}), \tau)$ be its extended Priestley dual. We define

$$D = \{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} = \mathfrak{x}^*\},$$

and let $\tau_\bowtie$ the topology on $I(\mathbf{A})$ induced as a subspace of $\mathcal{S}(\mathbf{A})$.

Lemma 7.1.5. *$(I(\mathbf{A}), \subseteq, D, \tau_\bowtie)$ is an unpointed Sugihara space.*

Proof. By Lemma 7.1.3 we have that $(I(\mathbf{A}), \subseteq, \tau_\bowtie)$ is an Esakia space. It suffices to prove that $(I(\mathbf{A}), \subseteq)$ is a forest and D is a clopen subset of $\subseteq$-minimal elements. The first of these demands is met since $\Omega_\mathbf{A}$ is an order isomorphism and $\mathcal{D}(\mathbf{A})$ is a forest.

For the second demand, note that $D \subseteq I(\mathbf{A})$ by Lemma 7.1.4(4). To see that each $\mathfrak{x} \in D$ is minimal, let $\mathfrak{y} \in I(\mathbf{A})$ such that $\mathfrak{y} \subseteq \mathfrak{x} = \mathfrak{x}^*$. This gives $e \in \mathfrak{y}$, and from * being antitone we obtain $\mathfrak{x} = \mathfrak{x}^* \subseteq \mathfrak{y}^*$. Thus $e \in \mathfrak{y}^*$. It follows that $e \in \mathfrak{y}, \mathfrak{y}^*$, so $\mathfrak{y} = \mathfrak{y}^*$ by Lemma 7.1.4. This implies that $\mathfrak{x} \subseteq \mathfrak{y} \subseteq \mathfrak{x}$, so $\mathfrak{x} = \mathfrak{y}$.

To prove D is clopen, note that $\mathfrak{x} \in D$ iff $\mathfrak{x} = \mathfrak{x}^*$ iff $e \in \mathfrak{x}$ and $\neg e \notin \mathfrak{x}$ iff $\mathfrak{x} \in \varphi(e) \cap \varphi(\neg e)^\mathsf{c}$. Since $D = \varphi(e) \cap \varphi(\neg e)^\mathsf{c}$ is a clopen subset of $\mathcal{S}(\mathbf{A})$, it is also clopen in the subspace $I(\mathbf{A})$. $\square$

Remark 7.1.6. It is easy to see that if $h \in \mathcal{D}(A)$ has its image contained in $\{-1, 1\}$, then setting $\mathfrak{x} = \Omega_\mathbf{A}(h)$ yields $\mathfrak{x} = \mathfrak{x}^*$. On the other hand, if $\mathfrak{x} = \mathfrak{x}^* \in \mathcal{S}(A)$, then by the surjectivity of $\Omega_\mathbf{A}$ there exists $h \in \mathcal{D}(A)$ with $\mathfrak{x} = \Omega_\mathbf{A}(h)$. Were there $a \in A$ with $h(a) = 0$, this would imply $h(\neg a) = 0$. Also, this would give that $a, \neg a \in \Omega_\mathbf{A}(h) = \mathfrak{x} = \mathfrak{x}^*$. But $a \in \mathfrak{x}^*$ gives $\neg a \notin \mathfrak{x}$, a contradiction. Therefore the image of h must lie in $\{-1, 1\}$, whence

$$\Omega_\mathbf{A}[\{h \in \mathcal{D}(A) : (\forall a \in A)(h(a) \in \{-1, 1\}\}] = \{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} = \mathfrak{x}^*\}.$$

It follows that $\Omega_{\mathbf{A}}$ preserves the designated subset D. Since $\Omega_{\mathbf{A}}$ is a bijection, this guarantees that it is an isomorphism in the category of unpointed Sugihara spaces.

The stage is set to describe the dual of the enriched negative cone functor.

Definition 7.1.7. *Given* $\mathbf{X} = (X, \leqslant, R, ^*, I, \tau)$ *an object of* $\mathsf{SM}_{\perp}^{\tau}$*, set*

$$X_{\bowtie} := I$$

$$D := \{x \in X : x = x^*\}$$

and let $\tau_{\bowtie}$ *be the topology on* $X_{\bowtie}$ *inherited as a subspace of* $\mathbf{X}$*. Define*

$$\mathbf{X}_{\bowtie} = (X_{\bowtie}, \leqslant, D, \tau_{\bowtie}.)$$

For a morphism $\alpha \colon \mathbf{X} \to \mathbf{Y}$ *of* $\mathsf{SM}_{\perp}^{\tau}$*, define* $\alpha_{\bowtie} = \alpha{\restriction}_{X_{\bowtie}}$*.*

Remark 7.1.8. In the previous definition, we overload the notation $(-)_{\bowtie}$ to provide a description of a construction on $\mathsf{SM}_{\perp}^{\tau}$. This anticipates that $(-)_{\bowtie}$ as defined above will provide a dual analogue of the enriched negative cone functor, and we use the same symbol by analogy (and may readily distinguish these uses by the type of the argument). When we introduce a dual analogue of the Galatos-Raftery construction in Section 7.2, we will make a similar use of $(-)^{\bowtie}$.

We now show that Definition 7.1.7 makes sense for objects, leaving an account of morphisms for Section 7.3.

Lemma 7.1.9. *Let* $\mathbf{X} = (X, \leqslant, R, ^*, I, \tau)$ *be an object of* $\mathsf{SM}_{\perp}^{\tau}$*. Then* $\mathbf{X}_{\bowtie}$ *is an unpointed Sugihara space.*

Proof. Extended Priestley duality implies that there exists $\mathbf{A} \in \mathsf{SM}_{\perp}$ with $\mathcal{S}(\mathbf{A}) \cong \mathbf{X}$ in $\mathsf{SM}_{\perp}^{\tau}$. Let $\alpha \colon \mathcal{S}(\mathbf{A}) \to \mathbf{X}$ be an isomorphism witnessing this. Then $\alpha[I(\mathbf{A})] = I$, and $\alpha\restriction_{I(\mathbf{A})}$ is a continuous order isomorphism. I inherits being a Hausdorff space from $\mathcal{S}(\mathbf{A})$, and since $I(\mathbf{A})$ is compact by Lemma 7.1.3 we have that $\varphi\restriction_{I(\mathbf{A})}$ is a homeomorphism. As before, we get that I is a Priestley space isomorphic to $I(\mathbf{A})$ and thus an Esakia space. That $(I, \leqslant)$ is a forest also follows from this order isomorphism, together with the fact that $(I(\mathbf{A}), \subseteq)$ is a forest by Lemma 7.1.5.

We need only prove that $D \subseteq I$ and that D is a clopen set of minimal elements. To this end, let $y \in D$. As α is a bijection, there exists $\mathfrak{x} \in \mathcal{S}(A)$ with $\alpha(\mathfrak{x}) = y$. We have $y = y^*$ from $y \in D$, whence $y^* = \alpha(\mathfrak{x})$. Since α preserves *, this implies $y = \varphi(\mathfrak{x}^*) = \varphi(\mathfrak{x})$. From α being one-to-one we get $\mathfrak{x}^* = \mathfrak{x}$, and

$$D \subseteq \alpha[\{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} = \mathfrak{x}^*\}].$$

As $\{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} = \mathfrak{x}^*\} \subseteq I(\mathbf{A})$ by Lemma 7.1.4(4), we have that $D \subseteq I$ as $\alpha[I(\mathbf{A})] = I$. Moreover, if $\mathfrak{x} = \mathfrak{x}^*$ in $\mathcal{S}(A)$, then $\alpha(\mathfrak{x}) = \alpha(\mathfrak{x}^*) = \alpha(\mathfrak{x})^*$ implies $\alpha(\mathfrak{x}) \in D$. Thus $\alpha[\{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} = \mathfrak{x}^*\}] \subseteq D$, whence $\alpha[\{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} = \mathfrak{x}^*\}] = D$. Since $\{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} = \mathfrak{x}^*\}$ is a clopen collection of minimal elements by Lemma 7.1.5, we infer that D is also a clopen collection of minimal elements of I (i.e., as α is an order isomorphism and homeomorphism). This means that $\mathbf{X}_{\bowtie} = (I, \leqslant, D, \tau_{\bowtie})$ is an unpointed Sugihara space, yielding the result. $\square$

7.2 Dual twist products

We now refocus our efforts to providing a dual presentation of $(-)^{\bowtie}$. This demands more detailed scrutiny of filter multiplication $\bullet$ in $\mathsf{SM}_{\perp}$. Note that $\bullet$ is a binary operation on $\mathcal{S}(A) \cup \{A\}$ for any bounded Sugihara monoid $\mathbf{A}$, as from Chapter 4, and we freely make use of the fact that $\bullet$ is associative, commutative, and order-preserving (cf. Lemma 4.1.5). For the following lemmas, let $\mathbf{A} \in \mathsf{SM}_{\perp}$.

Lemma 7.2.1. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A) \cup \{A\}$. Then the following hold.*

1. $\mathfrak{y} \in I(\mathbf{A})$ *implies* $\mathfrak{x} \subseteq \mathfrak{x} \bullet \mathfrak{y}$.

2. $\mathfrak{x} \bullet \mathfrak{x} = \mathfrak{x}$.

3. $ab \in \mathfrak{x}$ *implies* $a \in \mathfrak{x}$ *or* $b \in \mathfrak{x}$.

4. $a, b \in \mathfrak{x}$ *implies* $ab \in \mathfrak{x}$.

Proof. For (1), note that $a \in \mathfrak{x}$ implies $a = ae \in \mathfrak{x} \bullet \mathfrak{y}$, whence $\mathfrak{x} \subseteq \mathfrak{x} \bullet \mathfrak{y}$.

For (2), let $a \in \mathfrak{x}$. Then $a = a \cdot a \in \mathfrak{x} \bullet \mathfrak{x}$ since $\mathbf{A}$ is idempotent, and thus $\mathfrak{x} \subseteq \mathfrak{x} \bullet \mathfrak{x}$. For the reverse inclusion let $c \in \mathfrak{x} \bullet \mathfrak{x}$. Then there are $a, b \in \mathfrak{x}$ such that $ab \leqslant c$, whence $a \leqslant b \to c$. Thus $b \to c \in \mathfrak{x}$ from $\mathfrak{x}$ being an up-set. We have $b \wedge (b \to c) \leqslant b(b \to c) \leqslant c$, whence $c \in \mathfrak{x}$.

For (3), note that $ab \leqslant a \vee b$ in any bounded Sugihara monoids, and therefore the result follows from the primality of $\mathfrak{x}$.

For (4), use the fact that $a \wedge b \leqslant ab$ in any bounded Sugihara monoid. $\square$

Lemma 7.2.2. *Let $\mathfrak{x} \in \mathcal{S}(A)$. Then $\mathfrak{x} \wedge \mathfrak{x}^*$ exists, and moreover $\mathfrak{x} \wedge \mathfrak{x}^* = \mathfrak{x} \bullet \mathfrak{x}^*$.*

Proof. From Lemma 7.1.4(2) either $\mathfrak{x} \subseteq \mathfrak{x}^*$ or $\mathfrak{x}^* \subseteq \mathfrak{x}$, so the meet of $\mathfrak{x}$ and $\mathfrak{x}^*$ certainly exists.

For the rest, assume without loss of generality that $\mathfrak{x}^* \subseteq \mathfrak{x}$. Then $e \in \mathfrak{x}$, whence $\mathfrak{x}^* \subseteq \mathfrak{x}^* \bullet \mathfrak{x}$ by Lemma 7.2.1(1). For the reverse inclusion, let $c \in \mathfrak{x}^* \bullet \mathfrak{x}$. By definition there are $a \in \mathfrak{x}^*$ and $b \in \mathfrak{x}$ so that $ab \leqslant c$. The latter condition holds if and only if $a \cdot \neg c \leqslant \neg b$. Were it the case that $\neg c \in \mathfrak{x}$, then $b \cdot \neg c \leqslant \neg a$ would give $\neg a \in \mathfrak{x}$, a contradiction to $a \in \mathfrak{x}^*$. Hence $\neg c \notin \mathfrak{x}$, and consequently $c \in \mathfrak{x}^*$. This implies that $\mathfrak{x}^* \bullet \mathfrak{x} \subseteq \mathfrak{x}^*$, proving equality. $\square$

Lemma 7.2.3. *If $\mathfrak{x}, \mathfrak{y} \in I(\mathbf{A})$, then $\mathfrak{x} \vee \mathfrak{y}$ exists in $\mathcal{S}(A) \cup \{A\}$, and moreover $\mathfrak{x} \vee \mathfrak{y} = \mathfrak{x} \bullet \mathfrak{y}$.*

Proof. The hypothesis gives $\mathfrak{x}, \mathfrak{y} \subseteq x \bullet y$ by Lemma 7.2.1(1). Let $\mathfrak{z} \in \mathcal{S}(A) \cup \{A\}$ with $\mathfrak{x}, \mathfrak{y} \subseteq \mathfrak{z}$. The monotonicity of $\bullet$ gives $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z} \bullet \mathfrak{z} = \mathfrak{z}$, so $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} \vee \mathfrak{y}$ as claimed. $\square$

In the following, we use $\|$ to denote incomparability with respect to the order.

Lemma 7.2.4. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. If $\mathfrak{x} \| \mathfrak{y}$, then $\mathfrak{x} \vee \mathfrak{y}$ exists in $\mathcal{S}(A) \cup \{A\}$, and $\mathfrak{x} \vee y = \mathfrak{x} \bullet \mathfrak{y}$.*

Proof. Let $a \in \mathfrak{x} - \mathfrak{y}$ and $b \in \mathfrak{y} - \mathfrak{x}$. From $a \notin \mathfrak{y}$ we have $\neg a \in \mathfrak{y}^*$, and from $b \notin \mathfrak{x}$ we have $\neg b \in \mathfrak{x}^*$. Consequently, $a \cdot \neg a \in \mathfrak{x} \bullet \mathfrak{y}^*$ and $b \cdot \neg b \in \mathfrak{y} \bullet \mathfrak{x}^* = \mathfrak{x}^* \bullet \mathfrak{y}$. Also, $a \cdot \neg a = a \cdot (a \to \neg e) \leqslant \neg e \leqslant e$. Similarly, $b \cdot \neg b \leqslant \neg e \leqslant e$. It follows that $\neg e, e \in \mathfrak{x} \bullet \mathfrak{y}^*, \mathfrak{x}^* \bullet \mathfrak{y}$ since $\mathfrak{x} \bullet \mathfrak{y}^*$ and $\mathfrak{x}^* \bullet \mathfrak{y}$ are up-sets. There are four cases.

First, suppose $\mathfrak{x}, \mathfrak{y} \notin I(\mathbf{A})$. Then from Lemma 7.1.4 we have $\mathfrak{x} \subseteq \mathfrak{x}^*$ and $\mathfrak{y} \subseteq \mathfrak{y}^*$, whence $\mathfrak{x} \bullet \mathfrak{x}^* = \mathfrak{x}$ and $\mathfrak{y} \bullet \mathfrak{y}^* = \mathfrak{y}$ by Lemma 7.2.2. Since $e \in \mathfrak{x} \bullet \mathfrak{y}^*$, Lemma 7.2.1(1) implies $\mathfrak{y} \subseteq \mathfrak{x} \bullet \mathfrak{y}^* \bullet \mathfrak{y} = \mathfrak{x} \bullet \mathfrak{y}$. A similar argument gives $\mathfrak{x} \subseteq \mathfrak{x} \bullet \mathfrak{y}$, whence $\mathfrak{x}, \mathfrak{y} \subseteq \mathfrak{x} \bullet \mathfrak{y}$. Note that $\mathfrak{x}, \mathfrak{y} \subseteq \mathfrak{z}$, then $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$ follows from the monotonicity and idempotence of $\bullet$, and thus $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} \vee \mathfrak{y}$.

Second, suppose $\mathfrak{x} \notin I(\mathbf{A})$ and $\mathfrak{y} \in I(\mathbf{A})$. This implies $\mathfrak{x} \subseteq \mathfrak{x}^*$ and $\mathfrak{y}^* \subseteq \mathfrak{y}$, so from the latter $\mathfrak{x} \bullet \mathfrak{y}^* \subseteq \mathfrak{x} \bullet \mathfrak{y}$. It follows that $e \in \mathfrak{x} \bullet \mathfrak{y}$ as $e \in \mathfrak{x} \bullet \mathfrak{y}^*$. Hence $\mathfrak{x}, \mathfrak{y} \subseteq \mathfrak{x} \bullet \mathfrak{y}$, and $\mathfrak{x} \bullet \mathfrak{y}$ must be the least among upper bounds for the same reason as before.

The case for $\mathfrak{y} \notin I(\mathbf{A})$ and $\mathfrak{x} \in I(\mathbf{A})$ follows by symmetry. The case where $\mathfrak{x}, \mathfrak{y} \in I(\mathbf{A})$ follows from Lemma 7.2.3. $\square$

We caution that $\mathfrak{x} \vee \mathfrak{y}$ need not exist in $\mathcal{S}(A)$ in the previous two lemmas.

Lemma 7.2.5. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. If $\mathfrak{x} \subseteq \mathfrak{y} \subseteq \mathfrak{x}^*$, then $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x}$.*

Proof. The monotonicity and idempotence of $\bullet$ provides $\mathfrak{x} = \mathfrak{x} \bullet \mathfrak{x} \subseteq \mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{x} \bullet \mathfrak{x}^*$. But $\mathfrak{x} \bullet \mathfrak{x}^* = \mathfrak{x} \wedge \mathfrak{x}^* = \mathfrak{x}$ by Lemma 7.2.2, whence $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x}$. $\square$

Lemma 7.2.6. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. If $\mathfrak{x}$ and $\mathfrak{y}^*$ are comparable, then $\mathfrak{x}$ and $\mathfrak{y}$ are also comparable.*

Proof. Suppose that $\mathfrak{x}$ and $\mathfrak{y}^*$ are comparable. We suppose without loss of generality that $\mathfrak{x} \subseteq \mathfrak{y}^*$; the case where $\mathfrak{y}^* \subseteq \mathfrak{x}$ follows from exchanging the roles of $\mathfrak{x}$ and $\mathfrak{y}$ and the identity $\mathfrak{x} = \mathfrak{x}^{**}$. There are three cases.

<u>**Case 1:**</u> $\mathfrak{x} \in I(\mathbf{A})$. Then Lemma 7.1.4(3) and (4) provides that $\mathfrak{x}^* \subseteq \mathfrak{x}$, whence $\mathfrak{x}^* \subseteq \mathfrak{x} \subseteq \mathfrak{y}^*$. It follows that $\mathfrak{y} \subseteq \mathfrak{x}$.

<u>**Case 2:**</u> $\mathfrak{y}^* \notin I(\mathbf{A})$. Then from Lemma 7.1.4(3) we have $\mathfrak{y}^* \subseteq \mathfrak{y}$, whence from $\mathfrak{x} \subseteq \mathfrak{y}^*$ we get $\mathfrak{x} \subseteq \mathfrak{y}$.

<u>**Case 3:**</u> $\mathfrak{x} \notin I(\mathbf{A})$ and $\mathfrak{y}^* \in I(\mathbf{A})$. If $\mathfrak{y} \in I(\mathbf{A})$, then Lemma 7.1.4(4) gives that $\mathfrak{y} = \mathfrak{y}^*$ as $\mathfrak{y}, \mathfrak{y}^* \in I(\mathbf{A})$. We immediately get $\mathfrak{x} \subseteq \mathfrak{y}$ from this. Thus we assume that $\mathfrak{y} \notin I(\mathbf{A})$. Then $\mathfrak{x} \subset \mathfrak{x}^*$ and $\mathfrak{y} \subset \mathfrak{y}^*$, and by assumption $\mathfrak{x} \subset \mathfrak{y}^*$ and $\mathfrak{y} \subset \mathfrak{x}^*$. Then $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{x}^*, \mathfrak{y}^*$ follows from the monotonicity and idempotence of $\bullet$. Were it the case that $\mathfrak{x} \bullet \mathfrak{y} \in I(\mathbf{A})$, we would have $\mathfrak{x}^*, \mathfrak{y}^* \in \uparrow(\mathfrak{x} \bullet \mathfrak{y})$, the up-set considered in $I(\mathbf{A})$. From this, $\mathfrak{x}^*$ and $\mathfrak{y}^*$ are comparable since $I(\mathbf{A})$ is a forest, whence we get the comparability of $\mathfrak{x}$ and $\mathfrak{y}$. On the other hand, if $\mathfrak{x} \bullet y \notin I(\mathbf{A})$, then we argue toward a contradiction. If $\mathfrak{x}$ and $\mathfrak{y}$ are incomparable, then Lemma 7.2.4 implies $\mathfrak{x} \vee \mathfrak{y}$ exists and $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} \vee \mathfrak{y}$. Then $\mathfrak{x}, \mathfrak{y} \subseteq \mathfrak{x} \bullet \mathfrak{y}$, and if $\mathfrak{x} \bullet \mathfrak{y} \notin I(\mathbf{A})$ we have $\mathfrak{x}, \mathfrak{y} \in \downarrow(\mathfrak{x} \bullet \mathfrak{y})$ in the $*$-image of $I(\mathbf{A})$. But $*$ is a dual order isomorphism of $I(\mathbf{A})$ and $\{\mathfrak{z}^* : \mathfrak{z} \in I(\mathbf{A})\}$, so the $*$-image of $I(\mathbf{A})$ is a dual forest. This is a contradiction, and it follows that $\mathfrak{x}$ and $\mathfrak{y}$ are comparable. $\square$

Lemma 7.2.6 provides an important piece of information about the order of $\mathcal{S}(A)$, which is further developed in the following.

Corollary 7.2.7. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$ with $\mathfrak{x}$ and $\mathfrak{y}$ comparable. Then $\{\mathfrak{x}, \mathfrak{y}, \mathfrak{x}^*, \mathfrak{y}^*\}$ is a chain under subset inclusion.*

Proof. Each of the pairs $\mathfrak{x}$ and $\mathfrak{y}^*$ and $\mathfrak{x}^*$ and $\mathfrak{y}$ are comparable by Lemma 7.2.6. Any $\mathfrak{z} \in \mathcal{S}(A)$ is comparable to $\mathfrak{z}^*$ by Lemma 7.1.4(2), so it follows that $\mathfrak{x}^*$ and $\mathfrak{x}$ are comparable and $\mathfrak{y}^*$ and $\mathfrak{y}$ are comparable. Because x and y being comparable implies that $\mathfrak{x}^*$ and $\mathfrak{y}^*$ are comparable too, this means any two of $\mathfrak{x}, \mathfrak{y}, \mathfrak{x}^*, \mathfrak{y}^*$ are comparable. □

Lemma 7.2.8. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. If $\mathfrak{x} \notin I(\mathbf{A})$, $\mathfrak{y} \in I(\mathbf{A})$, $\mathfrak{x} \subseteq \mathfrak{y}$, and $\mathfrak{y} \nsubseteq \mathfrak{x}^*$, then $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{y}$.*

Proof. $\mathfrak{x}^*$ and $\mathfrak{y}$ are comparable by Corollary 7.2.7. Also, $\mathfrak{x}^* \subset \mathfrak{y}$ follows since $\mathfrak{y} \nsubseteq \mathfrak{x}^*$. This implies $\mathfrak{x} \subseteq \mathfrak{x}^* \subseteq \mathfrak{y}$, and from the monotonicity and idempotence of $\bullet$ we get $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{x}^* \bullet \mathfrak{y} \subseteq \mathfrak{y}$. As $\mathfrak{x}^* \subset \mathfrak{y}$, we get $\mathfrak{y}^* \subset \mathfrak{x}$. Let $a \in \mathfrak{x}$ with $a \notin \mathfrak{y}^*$. Then the second of these implies that $\neg a \in \mathfrak{y}$, so $a \cdot \neg a \in \mathfrak{x} \bullet \mathfrak{y}$. Thus since $a \cdot \neg a \leqslant e$, we get $e \in \mathfrak{x} \bullet \mathfrak{y}$ and consequently $\mathfrak{y} \subseteq \mathfrak{x} \bullet \mathfrak{y} \bullet \mathfrak{y} = \mathfrak{x} \bullet \mathfrak{y}$. This yields $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{y}$. □

Given $\mathbf{A} \in \mathsf{SM}_\perp$, we define the *absolute value* of $\mathfrak{x} \in \mathcal{S}(A)$ by

$$|\mathfrak{x}| = \mathfrak{x} \vee \mathfrak{x}^*.$$

Lemma 7.1.4 provides that the absolute value always exists, that $|\mathfrak{x}| = \mathfrak{x}$ or $|\mathfrak{x}| = \mathfrak{x}^*$, and that $|\mathfrak{x}| \in I(\mathbf{A})$.

Lemma 7.2.9. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. If $|\mathfrak{x}| \subset |\mathfrak{y}|$ and $\mathfrak{x} \subseteq \mathfrak{y}$, then $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{y}$.*

Proof. Notice that $|\mathfrak{y}| = \mathfrak{y}^*$ cannot occur: If $|\mathfrak{y}| = \mathfrak{y}^*$, then $|\mathfrak{x}| \subset |\mathfrak{y}|$ implies that $\mathfrak{x}^* \subseteq |\mathfrak{x}| \subset \mathfrak{y}^*$, and thus $\mathfrak{y} \subset \mathfrak{x}$. This is a contradiction to $\mathfrak{x} \subseteq \mathfrak{y}$. Hence $|\mathfrak{y}| = \mathfrak{y}$ from the definition of the absolute value. We consider two cases.

<u>Case 1:</u> $|\mathfrak{x}| = \mathfrak{x}$. Then $\mathfrak{x}, \mathfrak{y} \in I(\mathbf{A})$, and $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} \vee \mathfrak{y} = \mathfrak{y}$ from Lemma 7.2.3.

<u>Case 2:</u> $|\mathfrak{x}| = \mathfrak{x}^*$. If $\mathfrak{x} = \mathfrak{x}^*$, then Case 1 applies. Suppose that $\mathfrak{x} \neq \mathfrak{x}^*$, whence from Lemma 7.1.4(4) we have $\mathfrak{x} \notin I(\mathbf{A})$. Since $|\mathfrak{y}| = \mathfrak{y}$ by the remarks above, we have that $\mathfrak{y} \in I(\mathbf{A})$. The hypothesis gives $\mathfrak{x}^* \subset \mathfrak{y}$, so we have also that $\mathfrak{y} \nsubseteq \mathfrak{x}^*$. Thus $\mathfrak{x} \notin I(\mathbf{A})$, $\mathfrak{y} \in I(\mathbf{A})$, $\mathfrak{x} \subseteq \mathfrak{y}$, and $\mathfrak{y} \nsubseteq \mathfrak{x}^*$, and Lemma 7.2.8 implies that $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{y}$ as claimed. $\square$

Lemma 7.2.10. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. If $|\mathfrak{x}| \subset |\mathfrak{y}|$ and $\mathfrak{y} \subseteq \mathfrak{x}$, then $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{y}$.*

Proof. Note that $|\mathfrak{y}| \neq \mathfrak{y}$. To see this, observe that if $|\mathfrak{y}| = \mathfrak{y}$ then we would have $\mathfrak{x} \vee \mathfrak{x}^* = |\mathfrak{x}| \subset |\mathfrak{y}| = \mathfrak{y} \subseteq \mathfrak{x}$, a contradiction. Hence $|\mathfrak{y}| = \mathfrak{y}^*$, and

$$\mathfrak{y} \subseteq \mathfrak{x} \subseteq \mathfrak{x} \vee \mathfrak{x}^* = |\mathfrak{x}| \subset |\mathfrak{y}| = \mathfrak{y}^*.$$

Then $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{y}$ follows from Lemma 7.2.5. $\square$

Lemma 7.2.11. *Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. If $|\mathfrak{x}| = |\mathfrak{y}|$ and $\mathfrak{x} \subseteq \mathfrak{y}$, then $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} = \mathfrak{x} \wedge \mathfrak{y}$.*

Proof. From $|\mathfrak{x}| = |\mathfrak{y}|$ we have $\mathfrak{x} = \mathfrak{y}$ or $\mathfrak{x}^* = \mathfrak{y}$. If $\mathfrak{x} = \mathfrak{y}$, then $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} \bullet \mathfrak{x} = \mathfrak{x} = \mathfrak{x} \wedge \mathfrak{y}$ because $\bullet$ is idempotent. If $\mathfrak{x}^* = \mathfrak{y}$, then $\mathfrak{x} \subseteq \mathfrak{y} \subseteq \mathfrak{x}^*$, and $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} = \mathfrak{x} \wedge \mathfrak{y}$ follows from Lemma 7.2.5. $\square$

We have amassed enough information about $\bullet$ to offer a complete description. We summarize the results above in the following.

Lemma 7.2.12. *Let $\mathbf{A} \in \mathsf{SM}_\perp$ and let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$. We write $\mathfrak{x} \parallel \mathfrak{y}$ if $\mathfrak{x}$ and $\mathfrak{y}$ are incomparable, and $\mathfrak{x} \perp \mathfrak{y}$ if $\mathfrak{x}$ and $\mathfrak{y}$ are comparable. Then*

$$\mathfrak{x} \bullet \mathfrak{y} = \begin{cases} \mathfrak{x} \vee \mathfrak{y} & \text{if } \mathfrak{x}, \mathfrak{y} \in I(\mathbf{A}) \text{ or } \mathfrak{x} \parallel \mathfrak{y} \\ \mathfrak{y} & \text{if } \mathfrak{x} \perp \mathfrak{y} \text{ and } |\mathfrak{x}| \subset |\mathfrak{y}| \\ \mathfrak{x} & \text{if } \mathfrak{x} \perp \mathfrak{y} \text{ and } |\mathfrak{y}| \subset |\mathfrak{x}| \\ \mathfrak{x} \wedge \mathfrak{y} & \text{if } \mathfrak{x} \perp \mathfrak{y} \text{ and } |\mathfrak{x}| = |\mathfrak{y}| \end{cases}$$

where $\wedge$ and $\vee$ are evaluated in $\mathcal{S}(A) \cup \{A\}$.

Proof. Lemma 7.2.3 provides that $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} \vee \mathfrak{y}$ if $\mathfrak{x}, \mathfrak{y} \in I(\mathbf{A})$, and Lemma 7.2.4 provides $\mathfrak{x} \bullet \mathfrak{y} = \mathfrak{x} \vee \mathfrak{y}$ if $\mathfrak{x} \parallel \mathfrak{y}$.

In the remaining cases $\mathfrak{x} \perp \mathfrak{y}$ holds. If either $|\mathfrak{x}| \subset |\mathfrak{y}|$ or $|\mathfrak{y}| \subset |\mathfrak{x}|$, then Lemmas 7.2.9 and 7.2.10 show that $\mathfrak{x} \bullet \mathfrak{y}$ is whichever of $\mathfrak{x}$ or $\mathfrak{y}$ has the greater absolute value. If $\mathfrak{x} \perp \mathfrak{y}$ and $|\mathfrak{x}| = |\mathfrak{y}|$, then $\mathfrak{x} \cdot \mathfrak{y} = \mathfrak{x} \wedge \mathfrak{y}$ by Lemma 7.2.11. This proves the claim. $\square$

Remark 7.2.13. Corollary 7.2.7 implies that if $\mathfrak{x}$ and $\mathfrak{y}$ are comparable, then exactly one of $|x| \subset |y|$, $|x| = |y|$, or $|y| \subset |x|$ holds. This entails that Lemma 7.2.12 completely describes $\bullet$ for a Sugihara monoid $\mathbf{A}$.

Remark 7.2.14. Compare Lemma 7.2.12 with the definition of $\cdot$ on the Sugihara monoids $\mathbf{S}$ and $\mathbf{S}\backslash\{0\}$ (see Examples 2.3.8 and 2.3.9), which generate SM as a quasivariety by Proposition 2.3.12.

We will now construct our dual analogue $(-)^{\bowtie}$ on the level of objects. Let $\mathbf{X} = (X, \leqslant, D, \tau)$ be an unpointed Sugihara space, and let $-D^{\mathsf{c}} = \{-x : x \in D^{\mathsf{c}}\}$ be a formal copy of D^{c} disjoint from X. Set

$$X^{\bowtie} := X \cup -D^{\mathsf{c}}.$$

We extend $-$ to give a unary operation on $X^{\bowtie}$ by defining $-(-x) = x$ for $-x \in -D^{\mathsf{c}}$, and $-x = x$ for $x \in D$. We also define a partial order $\leqslant^{\bowtie}$ on $X^{\bowtie}$ by

1. If $x, y \in X$, then $x \leqslant^{\bowtie} y$ if and only if $x \leqslant y$,

2. If $-x, -y \in -D^{\mathsf{c}}$, then $-x \leqslant^{\bowtie} -y$ if and only if $y \leqslant x$,

3. If $-x \in -D^{\mathsf{c}}$ and $y \in X$, then $-x \leqslant^{\bowtie} y$ if and only if x and y are $\leqslant$-comparable.

For each $\mathbf{A} \in \mathsf{SM}_{\perp}$, define $\Gamma_{\mathbf{A}} \colon \mathcal{S}(A) \to I(\mathbf{A})^{\bowtie}$ by

$$\Gamma_{\mathbf{A}}(\mathfrak{x}) = \begin{cases} \mathfrak{x} & \text{if } \mathfrak{x} \in I(\mathbf{A}) \\ -(\mathfrak{x}^*) & \text{if } \mathfrak{x} \notin I(\mathbf{A}) \end{cases}$$

According to Lemma 7.1.4, one of $\mathfrak{x} \in I(\mathbf{A})$ or $\mathfrak{x}^* \in I(\mathbf{A})$ holds for all $\mathfrak{x} \in \mathcal{S}(A)$, and moreover $\mathfrak{x} = \mathfrak{x}^* = -\mathfrak{x}$ if both hold. This yields that $\Gamma_{\mathbf{A}}$ is well-defined.

Lemma 7.2.15. *$\Gamma_{\mathbf{A}}$ is an order isomorphism.*

Proof. We first prove that $\Gamma_{\mathbf{A}}$ is isotone, so let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$ with $\mathfrak{x} \subseteq \mathfrak{y}$. If $\mathfrak{x}, \mathfrak{y} \in I(\mathbf{A})$, then the result is obvious. If $\mathfrak{x}, \mathfrak{y} \notin I(\mathbf{A})$, then $\Gamma_{\mathbf{A}}(\mathfrak{x}) = -(\mathfrak{x}^*) \leqslant^{\bowtie} -(\mathfrak{y}^*) = \Gamma_{\mathbf{A}}(\mathfrak{y})$ from $\mathfrak{y}^* \subseteq \mathfrak{x}^*$. If $\mathfrak{x} \notin I(\mathbf{A})$ and $\mathfrak{y} \in I(\mathbf{A})$, then there is $\mathfrak{z} \in I(\mathbf{A})$ with $\mathfrak{x} = \mathfrak{z}^*$. As $\mathfrak{x}$ and $\mathfrak{y}$ are $\subseteq$-comparable, we get that $\mathfrak{y}$ and $\mathfrak{x}^* = \mathfrak{z}$ are comparable as well. Then $-\mathfrak{z} \leqslant^{\bowtie} \mathfrak{y}$ gives $\Gamma_{\mathbf{A}}(\mathfrak{x}) \leqslant^{\bowtie} \Gamma_{\mathbf{A}}(\mathfrak{y})$.

Second, we prove that $\Gamma_{\mathbf{A}}$ reflects the order. Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}(A)$ be such that $\Gamma_{\mathbf{A}}(\mathfrak{x}) \leqslant^{\bowtie} \Gamma_{\mathbf{A}}(\mathfrak{y})$. If $\mathfrak{x}, \mathfrak{y} \in I(\mathbf{A})$, then $\mathfrak{x} \subseteq \mathfrak{y}$ follows immediately. If $\mathfrak{x}, \mathfrak{y} \notin I(\mathbf{A})$, then we have that there are $\mathfrak{u}, \mathfrak{v} \in I(\mathbf{A})$ with $\mathfrak{x} = \mathfrak{u}^*$ and $\mathfrak{y} = \mathfrak{v}^*$ and $\Gamma_{\mathbf{A}}(\mathfrak{x}) = -\mathfrak{u}$ and $\Gamma_{\mathbf{A}}(\mathfrak{y}) = -\mathfrak{v}$. This gives $-\mathfrak{u} \leqslant^{\bowtie} -\mathfrak{v}$. By definition, the latter holds if and only if $\mathfrak{v} \subseteq \mathfrak{u}$, whence $\mathfrak{x} = \mathfrak{u}^* \subseteq \mathfrak{v}^* = \mathfrak{y}$. In the final case, suppose that $\mathfrak{x} \notin I(\mathbf{A})$ and $\mathfrak{y} \in I(\mathbf{A})$. Then there is $\mathfrak{u} \in I(\mathbf{A})$ such that $\mathfrak{x} = \mathfrak{u}^*$, and we have $\Gamma_{\mathbf{A}}(\mathfrak{x}) = -\mathfrak{u}$, $\Gamma_{\mathbf{A}}(\mathfrak{y}) = \mathfrak{y}$. By definition, $-\mathfrak{u} \leqslant^{\bowtie} \mathfrak{y}$ if and only if $\mathfrak{u}$ and $\mathfrak{y}$ are $\subseteq$-comparable. If $\mathfrak{u} \subseteq \mathfrak{y}$, then $\mathfrak{u}^* \subseteq \mathfrak{u} \subseteq \mathfrak{y}$ provides that $\mathfrak{x} \subseteq \mathfrak{y}$. If $\mathfrak{y} \subseteq \mathfrak{u}$, then $\mathfrak{x} = \mathfrak{u}^* \subseteq \mathfrak{y}^* \subseteq \mathfrak{y}$ gives the result. Hence $\Gamma_{\mathbf{A}}$ is order-reflecting.

Third and finally, we prove $\Gamma_{\mathbf{A}}$ surjective. Let $x \in I(\mathbf{A})^{\bowtie}$. If $x \in I(\mathbf{A})$, then $\Gamma_{\mathbf{A}}(x) = x$. If $x \notin I(\mathbf{A})$, then there is $y \in I(\mathbf{A})$ such that $x = -y$. Then $\Gamma_{\mathbf{A}}(y^*) = -y = x$, which proves the claim. □

Lemma 7.2.16. *Let $\mathbf{A} \in \mathsf{SM}_{\perp}$ and let $\mathfrak{x} \in \mathcal{S}(A)$. Then $\Gamma_{\mathbf{A}}(\mathfrak{x}^*) = -\Gamma_{\mathbf{A}}(\mathfrak{x})$.*

Proof. If $\mathfrak{x} \in I(\mathbf{A})$ and $\mathfrak{x}^* \notin I(\mathbf{A})$, we get that $\Gamma_{\mathbf{A}}(\mathfrak{x}^*) = -(\mathfrak{x}^{**}) = -\mathfrak{x} = \Gamma_{\mathbf{A}}(\mathfrak{x})$. If $\mathfrak{x}, \mathfrak{x}^* \in I(\mathbf{A})$, then Lemma 7.1.4 yields that $\mathfrak{x} = \mathfrak{x}'$, whence $\Gamma_{\mathbf{A}}(\mathfrak{x}^*) = \mathfrak{x}^* = \mathfrak{x} = \Gamma_{\mathbf{A}}(\mathfrak{x})$. In the last case, if $\mathfrak{x} \notin I(\mathbf{A})$ and $\mathfrak{x}^* \in I(\mathbf{A})$, then $\Gamma_{\mathbf{A}}(\mathfrak{x}^*) = \mathfrak{x}^* = -(-(\mathfrak{x}^*)) = -\Gamma_{\mathbf{A}}(\mathfrak{x})$. The claim hence holds in all cases, which settles the proof. $\square$

Lemmas 7.2.15 and 7.2.16 provide that $(\mathcal{S}(A), \subseteq, ')$ and $(I(\mathbf{A}), \subseteq^{\bowtie}, -)$ are isomorphic structures for any $\mathbf{A} \in \mathsf{SM}_{\perp}$. Keeping with our by-now-familiar *modus operandi*, we enrich these structures in order to expand the structure-preserving properties of $\Gamma_{\mathbf{A}}$. Let $\tau^{\bowtie}$ be the disjoint union topology on $X \cup -D^{\mathsf{c}}$, where the topology on $-D^{\mathsf{c}}$ is comes from considering it as a (copy of a) subspace of $\mathbf{X}$.

Lemma 7.2.17. *When $I(\mathbf{A})^{\bowtie}$ is given the topology $\tau^{\bowtie}$, $\Gamma_{\mathbf{A}}$ is continuous.*

Proof. Let $U \subseteq I(\mathbf{A})$ and $V \subseteq -\{\mathfrak{x} \in I(\mathbf{A}) : \mathfrak{x} = \mathfrak{x}^*\}^{\mathsf{c}}$ be open. Notice that U is an open subset of a clopen subspace of $\mathcal{S}(\mathbf{A})$, whence U is open in $\mathcal{S}(\mathbf{A})$. Also, the definition of V being open in $-\{\mathfrak{x} \in I(\mathbf{A}) : \mathfrak{x} = \mathfrak{x}^*\}^{\mathsf{c}}$ gives exactly that $\{\mathfrak{x} \in I(\mathbf{A}) : -\mathfrak{x} \in V\}$ is open in the clopen subspace $\{\mathfrak{x} \in I(\mathbf{A}) : \mathfrak{x} \neq \mathfrak{x}^*\}$ of $\mathcal{S}(\mathbf{A})$, and hence is open in $\mathcal{S}(\mathbf{A})$ too. Note that ${}^*\colon \mathcal{S}(A) \to \mathcal{S}(A)$ is continuous, whence inverse image $\{\mathfrak{x}^* : -\mathfrak{x} \in V\}$ of $\{\mathfrak{x} \in I(\mathbf{A}) : -\mathfrak{x} \in V\}$ under * is open in $\mathcal{S}(\mathbf{A})$. This implies

$$\begin{aligned}\Gamma_{\mathbf{A}}^{-1}[U \cup V] &= \Gamma_{\mathbf{A}}^{-1}[U] \cup \Gamma_{\mathbf{A}}^{-1}[V] \\ &= U \cup \{\mathfrak{x}^* \in \mathcal{S}(A) : -\mathfrak{x} \in V\}\end{aligned}$$

is open. Because an arbitrary $\tau^{\bowtie}$-open set has the form $U \cup V$ for U and V as above, the result follows. $\square$

Lemma 7.2.18. *Let $(X, \leqslant, D, \tau)$ be an object of SS. Then $(X^{\bowtie}, \tau^{\bowtie})$ is a compact Hausdorff space.*

Proof. D is clopen, so D^{c} is a closed subspace of the compact Hausdorff space (X, τ). This implies that $-D^{\mathsf{c}}$ (being a copy of D^{c}) is a compact Hausdorff space. Because $(X^{\bowtie}, \tau^{\bowtie})$ is a disjoint union of two compact Hausdorff spaces, the claim is proven. $\square$

Lemma 7.2.19. $\Gamma_{\mathbf{A}}$ *is a homeomorphism.*

Proof. Lemma 7.1.5 gives us that $(I(\mathbf{A}), \subseteq, D, \tau)$ is an object of SS, where as usual $D = \{\mathfrak{x} \in I(\mathbf{A}) : \mathfrak{x} = \mathfrak{x}^*\}$ and τ is the subspace topology coming from $\mathcal{S}(\mathbf{A})$. This implies $I(\mathbf{A})^{\bowtie}$ is a compact Hausdorff space by Lemma 7.2.18. Because $\mathcal{S}(\mathbf{A})$ is also compact, $\Gamma_{\mathbf{A}}$ is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism. $\square$

Take an object $\mathbf{X} = (X, \leqslant, D, \tau)$ of SS, and let $\mathbf{A} \in \mathsf{SM}_{\perp}$ with $\mathcal{D}(\mathbf{A}) \cong \mathbf{X}$. As a consequence of Remark 7.1.6 we have

$$\mathbf{X} \cong \mathcal{D}(\mathbf{A}) \cong I(\mathbf{A}).$$

and hence

$$(X^{\bowtie}, \leqslant^{\bowtie}, -) \cong (I(\mathbf{A})^{\bowtie}, \subseteq^{\bowtie}, -) \cong (\mathcal{S}(A), \subseteq, ^{*}),$$

where the last isomorphism is witnessed by $\Gamma_{\mathbf{A}}$. Note that for any $\mathbf{A}$, the partial operation $\bullet$ on $\mathcal{S}(\mathbf{A})$ is completely determined by the order and the involution by Lemma 7.2.12. This means that for each object $\mathbf{X} = (X, \leqslant, D, \tau)$ of SS we may

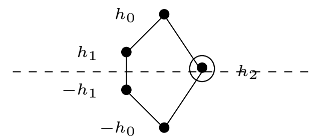


Figure 7.1: Labeled Hasse diagram for $\mathcal{D}(\mathbf{E})^{\bowtie}$

define a partial multiplication $\bullet$ on $X^{\bowtie}$ by

$$x \bullet y = \begin{cases} x \vee y & \text{if } x, y \in X \text{ or } x \parallel y \text{, provided the join exists} \\ z & \text{if } x \perp y, |y| \neq |x|, z \in \{x, y\}, \text{ and } |z| = \max\{|x|, |y|\} \\ x \wedge y & \text{if } x \perp y \text{ and } |x| = |y| \\ \text{undefined} & \text{otherwise} \end{cases}$$

where $|x| = x$ if $x \in X$, and $|-x| = x$ if $-x \in -D^{\mathsf{c}}$. We can also define a ternary relation R on $X^{\bowtie}$ by $R(x, y, z)$ if and only if $x \bullet y$ exists and $x \bullet y \leqslant^{\bowtie} z$. This is the last ingredient needed to define our dual analogue of $(-)^{\bowtie}$.

Definition 7.2.20. *For an unpointed Sugihara space $\mathbf{X} = (X, \leqslant, D, \tau)$, let $X^{\bowtie}$, $\leqslant^{\bowtie}$, $-$, R, and $\tau^{\bowtie}$ be as above. Define $\mathbf{X}^{\bowtie} = (X^{\bowtie}, \leqslant^{\bowtie}, R, -, X, \tau^{\bowtie})$. Given a morphism $\alpha\colon (X, \leqslant_X, D_X, \tau_X) \to (Y, \leqslant_Y, D_Y, \tau_Y)$ of SS, define $\alpha^{\bowtie}\colon \mathbf{X}^{\bowtie} \to \mathbf{Y}^{\bowtie}$ by*

$$\alpha^{\bowtie}(x) = \begin{cases} \alpha(x) & \text{if } x \in X, \\ -\alpha(-x) & \text{if } x \in -D_X^{\mathsf{c}} \end{cases}$$

Before we prove that Definition 7.2.20 makes sense on the level of objects, we offer an example to build intuition.

Example 7.2.21. *In Example 6.2.33, we introduced the bounded expansion $\mathbf{E}_{\perp}$ of the Sugihara monoid $\mathbf{E}$ (which was first described in Example 2.3.11). Figure*

7.2.21 gives the result of applying the construction of Definition 7.2.20 to $\mathcal{D}(\mathbf{E}_\perp)$. Unlike its algebraic counterpart, the dual version of $(-)^{\bowtie}$ is pictorial: It proceeds by copying each element in $\mathcal{D}(\mathbf{E}_\perp)$ besides h_2 (which is the sole element of the designated subset), and reflecting the copied points across the axis determined by the designated subset. The fact that the copied elements are reflected "below" the aforementioned axis motivates our decoration of the copied elements with $-$. It is easy to verify that $\mathcal{D}(\mathbf{E}_\perp)^{\bowtie}$ and $\mathcal{S}(\mathbf{E}_\perp)$ are isomorphic.

The next lemma establishes that Definition 7.2.20 makes sense for objects.

Lemma 7.2.22. *Let $\mathbf{X} = (X, \leqslant, D, \tau)$ be an unpointed Sugihara space. Then $\mathbf{X}^{\bowtie}$ is an object of $\mathsf{SM}_\perp^\tau$.*

Proof. Because SS and $\mathsf{SM}_\perp$ are dually-equivalent there exists $\mathbf{A} \in \mathsf{SM}_\perp$ such that $\mathbf{X} \cong \mathcal{D}(\mathbf{A})$ in SS. This observation and Remark 7.1.6 gives that, via $\Omega_{\mathbf{A}}$,

$$\mathbf{X} \cong \mathcal{D}(\mathbf{A}) \cong (I(\mathbf{A}), \subseteq, D_I, \tau_I),$$

where as before $D_I = \{\mathfrak{x} \in \mathcal{S}(A) : \mathfrak{x} = \mathfrak{x}^*\}$ and τ_I is the topology that $I(\mathbf{A})$ inherits as a subspace of $\mathcal{S}(\mathbf{A})$. Thus in SS we have

$$\mathbf{X} \cong (I(\mathbf{A}), \subseteq, D_I, \tau_I).$$

Note that there is a map $\alpha\colon (X^{\bowtie}, \leqslant^{\bowtie}, -, \tau^{\bowtie}) \to (I(\mathbf{A})^{\bowtie}, \subseteq^{\bowtie}, -, \tau_I^{\bowtie})$ that is an order isomorphism, homeomorphism, and preserves $-$. Also, $\Gamma_{\mathbf{A}}\colon \mathcal{S}(A) \to I(\mathbf{A})^{\bowtie}$ is an order isomorphism (by Lemma 7.2.15), a homeomorphism (by 7.2.19), and preserves the involution (by Lemma 7.2.16). This implies that $\delta := \Gamma_{\mathbf{A}}^{-1} \circ \varphi$ is an order isomorphism, homeomorphism, and preserves the involution. As $\mathcal{S}(\mathbf{A})$ is an object of $\mathsf{SM}^{\tau}_{\perp}$, it is enough to show that $\delta[X] = I(\mathbf{A})$ and that δ is an isomorphism with respect to R (i.e., for all $x, y, z \in X^{\bowtie}$, $R(x, y, z)$ if and only if $R(\delta(x), \delta(y), \delta(z))$).

From the fact that both $\Gamma_{\mathbf{A}}$ and α are bijections,

$$\delta[X] = (\Gamma_{\mathbf{A}}^{-1} \circ \alpha)[X] = \Gamma^{-1}[I(\mathbf{A})] = I(\mathbf{A})$$

To see that δ is an isomorphism with respect to R, let $x, y, z \in X^{\bowtie}$. Note that δ preserves the involution and preserves and reflects the order, whence because $\bullet$ is characterized entirely in terms of the involution and order we have that the following are equivalent

- $x \bullet y$ exists and $x \bullet y \leqslant^{\bowtie} x$.
- $\delta(z) \in \mathcal{S}(A)$ and $\delta(x) \bullet \delta(y) \subseteq \delta(z)$.

Hence $R(x, y, z)$ if and only if $R(\delta(x), \delta(y), \delta(z))$ as desired. Thus $\mathbf{X}^{\bowtie}$ is an object of $\mathsf{SM}^{\tau}_{\perp}$ and is isomorphic in that category to $\mathcal{S}(\mathbf{A})$. □

7.3 An equivalence between SS and $\mathsf{SM}^{\tau}_{\perp}$

In this final section of the chapter, we attend to categorical details. Although the primary interest in the dual variants of $(-)_{\bowtie}$ and $(-)^{\bowtie}$ arises from the representations they give us for objects, we may also describe the action of these constructions on morphisms and show that they give the functors of a categorical equivalence. Our first goal is to verify that Definitions 7.1.7 and 7.2.20 make sense for morphisms.

Lemma 7.3.1. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of $\mathsf{SM}^{\tau}_{\perp}$. Then $\alpha_{\bowtie}$ is a morphism of SS.*

Proof. Since α is a bounded morphism, we have $\alpha^{-1}[Y_{\bowtie}] = X_{\bowtie}$. This yields $\alpha[X_{\bowtie}] = \alpha[\alpha^{-1}[Y_{\bowtie}]] \subseteq Y_{\bowtie}$, and it follows that $\alpha{\restriction}_{X_{\bowtie}}$ has its image in $Y_{\bowtie}$. This means $\alpha_{\bowtie}$ is well-defined.

$\alpha_{\bowtie}$ is the restriction of a continuous isotone map, hence is itself a continuous isotone map. To prove that $\alpha_{\bowtie}$ is an Esakia map, let $x \in X_{\bowtie}$, $z \in Y_{\bowtie}$ such that $\alpha_{\bowtie}(x) \leqslant z$. Then as $\alpha(x), z \in Y_{\bowtie}$, from the definition of $\bullet$ we get that $\alpha(x) \bullet z = \alpha(x) \vee z = z$. This gives $R_Y \alpha(x)zz$. Because α is a bounded morphism, there hence are $u, v \in X$ with $R_X xuv$, $z \leqslant \alpha(u)$, and $\alpha(v) \leqslant z$. From $z \leqslant \alpha(u)$ and $z \in Y_{\bowtie}$ we obtain that $\alpha(u) \in Y_{\bowtie}$. Applying that α is a bounded morphism again, we have that $\alpha(u) \in Y_{\bowtie}$ implies that $u \in \alpha^{-1}[Y_{\bowtie}] = X_{\bowtie}$. The definition of $\bullet$ and $x, u \in X_{\bowtie}$ provide that $x \bullet u = x \vee u$. But $R_X xuv$ yields $x \bullet u \leqslant v$, whence $x, u \leqslant x \vee u \leqslant v$. By monotonicity we obtain $\alpha(v) \leqslant z \leqslant \alpha(u) \leqslant \alpha(v)$, and thus $x \leqslant v$ and $z = \alpha(v)$. Thus $\alpha_{\bowtie}$ is an Esakia function.

For the rest, observe that if $x \in X$ and $x^* = x$, then $\alpha_{\bowtie}(x) = \alpha_{\bowtie}(x)^*$ since α preserves *. Also, if $x \neq x^*$, then without loss of generality $x \in X_{\bowtie}$ and $x^* \notin X_{\bowtie} = \alpha^{-1}[Y_{\bowtie}]$, whence $\alpha(x) \in Y_{\bowtie}$ and $\alpha(x^*) \notin Y_{\bowtie}$. This implies $\alpha(x) \neq \alpha(x)^*$, proving the claim. $\square$

The proof that $\alpha^{\bowtie}$ is a bounded morphism for each morphism of SS is complicated, and involves some case analysis. For clarity of exposition, we divide the proof into several lemmas.

Lemma 7.3.2. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of SS. Then $\alpha^{\bowtie}$ is isotone.*

Proof. Let $x, y \in X^{\bowtie}$ with $x \leqslant^{\bowtie} y$.

Case 1: $x, y \in X$. In this case, $\alpha^{\bowtie}(x) = \alpha(x) \leqslant \alpha(y) = \alpha^{\bowtie}(y)$ follows because α is isotone.

Case 2: $x, y \notin X$. Here $x \leqslant^{\bowtie} y$ implies $-y \leqslant -x$, and from the isotonicity of α we obtain $-\alpha^{\bowtie}(y) = \alpha(-y) \leqslant \alpha(-x) = -\alpha^{\bowtie}(x)$. Thus $\alpha^{\bowtie}(x) \leqslant^{\bowtie} \alpha^{\bowtie}(y)$.

Case 3: $x \notin X$ and $y \in X$. For this case, $x \notin X$ gives that $-x \in X$, and $x \leqslant^{\bowtie} y$ implies $-x$ and y are $\leqslant$-comparable. Since α is isotone, this gives that $-\alpha^{\bowtie}(x) = \alpha(-x)$ and $\alpha^{\bowtie}(y) = \alpha(y)$ are $\leqslant$-comparable. The definition of $\alpha^{\bowtie}$ and the fact that $x \notin X$ imply that $\alpha^{\bowtie}(x) \notin Y$, whence from the definition of $\leqslant^{\bowtie}$ we get $\alpha^{\bowtie}(x) \leqslant^{\bowtie} \alpha^{\bowtie}(y)$. This settles the claim. □

Lemma 7.3.3. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of SS. Then $\alpha^{\bowtie}(-x) = -\alpha^{\bowtie}(x)$ for all $x \in X^{\bowtie}$.*

Proof. There are three cases.

Case 1: $x \in X \backslash D_X$. Here we have that $-x \in -D_X^{\mathsf{c}}$, and this gives that $\alpha^{\bowtie}(-x) = -\alpha(-(-x)) = -\alpha(x) = -\alpha^{\bowtie}(x)$.

Case 2: $x \in D_X$. In this situation, we have $\alpha^{\bowtie}(-x) = \alpha^{\bowtie}(x) = -\alpha^{\bowtie}(x)$.

Case 3: $x \in -D_X^{\mathsf{c}}$. We have that $-x \in X \backslash D_X$, and from this we obtain that $\alpha^{\bowtie}(-x) = \alpha(-x) = -(-\alpha(-x)) = -\alpha^{\bowtie}(x)$. □

Lemma 7.3.4. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of SS. Then $\alpha^{\bowtie}(|x|) = |\alpha^{\bowtie}(x)|$ for each $x \in X^{\bowtie}$.*

Proof. Let $x \in X^{\bowtie}$, and note that one of $-x \leqslant^{\bowtie} x$ or $x \leqslant^{\bowtie} -x$ holds. As $\alpha^{\bowtie}$ preserves $\leqslant^{\bowtie}$ by Lemma 7.3.2 and preserves $-$ by Lemma 7.3.3, we get $-\alpha^{\bowtie}(x) \leqslant^{\bowtie} \alpha^{\bowtie}(x)$ in the first case. In the second case, we obtain $\alpha^{\bowtie}(x) \leqslant^{\bowtie} -\alpha^{\bowtie}(x)$. Thus either $\alpha^{\bowtie}(x) \vee -\alpha^{\bowtie}(x) = \alpha^{\bowtie}(x) = \alpha^{\bowtie}(|x|)$ (in the first case), or else

$$\alpha^{\bowtie}(x) \vee -\alpha^{\bowtie}(x) = -\alpha^{\bowtie}(x) = \alpha^{\bowtie}(-x) = \alpha^{\bowtie}(|x|)$$

(in the second case). □

Lemma 7.3.5. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of SS. Then $\alpha^{\bowtie}$ preserves the ternary relation R.*

Proof. Let $x, y, z \in X^{\bowtie}$ such that $R_X(x, y, z)$. Then by definition $x \bullet y$ exists and $x \bullet y \leqslant^{\bowtie} z$. There are two cases.

Case 1: $x \bullet y = x \vee y$. In this situation, $x \vee y \leqslant^{\bowtie} z$, so $x \leqslant^{\bowtie} z$ and $y \leqslant^{\bowtie} z$. As $\alpha^{\bowtie}$ preserves the order, $\alpha^{\bowtie}(x), \alpha^{\bowtie}(y) \leqslant^{\bowtie} \alpha^{\bowtie}(z)$. Since $\bullet$ is order-preserving and idempotent, this implies $\alpha^{\bowtie}(x) \bullet \alpha^{\bowtie}(y) \leqslant^{\bowtie} \alpha^{\bowtie}(z)$. Therefore $R_Y(\alpha^{\bowtie}(x), \alpha^{\bowtie}(y), \alpha^{\bowtie}(z))$.

Case 2: $x \bullet y \neq x \vee y$. By the definition $\bullet$, we have $x \bullet y$ is one of x or y, and also $x \perp y$. Suppose without loss of generality that $x \leqslant^{\bowtie} y$ and (since $x \bullet y \neq x \vee y$) that $x \bullet y = x$. Then $|y| \leqslant^{\bowtie} |x|$ from the definition of $\bullet$. From Lemma 7.3.2 we get $\alpha^{\bowtie}(x) \leqslant^{\bowtie} \alpha^{\bowtie}(y)$, whence $\alpha^{\bowtie}(x) \bullet \alpha^{\bowtie}(y)$ must exist by the definition of $\bullet$. Moreover, $|y| \leqslant^{\bowtie} |x|$ together with Lemmas 7.3.2 and 7.3.4 yields $|\alpha^{\bowtie}(y)| \leqslant^{\bowtie} |\alpha^{\bowtie}(x)|$. Thus $\alpha^{\bowtie}(x) \bullet \alpha^{\bowtie}(y)$ is either $\alpha^{\bowtie}(x) \wedge \alpha^{\bowtie}(y)$ or whichever of $\alpha^{\bowtie}(x)$ and $\alpha^{\bowtie}(y)$ has greater absolute value by the definition of $\bullet$. This implies $\alpha^{\bowtie}(x) \bullet \alpha^{\bowtie}(y) = \alpha^{\bowtie}(x)$ in either case. Since $x = x \bullet y \leqslant^{\bowtie} z$, we get $\alpha^{\bowtie}(x) \bullet \alpha^{\bowtie}(y) = \alpha^{\bowtie}(x) \leqslant^{\bowtie} \alpha^{\bowtie}(z)$, and thus $R_Y(\alpha^{\bowtie}(x), \alpha^{\bowtie}(y), \alpha^{\bowtie}(z))$. ☐

Lemma 7.3.6. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of SS. Then if $R_Y(x, y, \alpha^{\bowtie}(z))$, there exists $u, v \in X^{\bowtie}$ such that $R_X(u, v, z)$, $x \leqslant^{\bowtie} \alpha^{\bowtie}(u)$, and $y \leqslant^{\bowtie} \alpha^{\bowtie}(v)$.*

Proof. Suppose that $R_Y(x, y, \alpha^{\bowtie}(z))$. By definition $x \bullet y$ exists and $x \bullet y \leqslant^{\bowtie} \alpha^{\bowtie}(z)$, and there are two possibilities.

Case 1: $x \bullet y = x \vee y$. Here $x \leqslant^{\bowtie} \alpha^{\bowtie}(z)$ and $y \leqslant^{\bowtie} \alpha^{\bowtie}(z)$. Taking $u = v = z$ gives the claim as $R_X(z, z, z)$.

Case 2: $x \bullet y \neq x \vee y$. Then from the definition of $\bullet$ we have that $x \perp y$ and $x \bullet y$ is one of x or y. Suppose without loss of generality that $x \leqslant^{\bowtie} y$, that $x \bullet y = x$ (for if $x \bullet y = y$, then we obtain the contradiction $x \bullet y = x \vee y$), and that $|y| \leqslant^{\bowtie} |x|$. Were it the case that $x, y \in Y$, we would have $x \bullet y = x \vee y$ by the definition of $\bullet$. Thus we may further suppose that $x \notin Y$, whence $|x| = -x$ (for otherwise $x \leqslant^{\bowtie} y$ and Y being an up-set would give $x, y \in Y$). The hypothesis that $x = x \bullet y \leqslant^{\bowtie} \alpha^{\bowtie}(z)$ implies

$\alpha^{\bowtie}(-z) \leqslant^{\bowtie} -x$. Therefore $\alpha^{\bowtie}(|z|)$ must be comparable to $-x$ by Corollary 7.2.7 (as transferred along the obvious isomorphism). This means either $\alpha^{\bowtie}(|z|) \leqslant^{\bowtie} -x$ or $-x \leqslant^{\bowtie} \alpha^{\bowtie}(|z|)$.

<u>Subcase 2.1:</u> $\alpha^{\bowtie}(|z|) \leqslant^{\bowtie} -x$. In this setting, $\alpha(|z|) \leqslant -x$ and α being an Esakia map provides that there are $u \in X$ such that $|z| \leqslant u$ and $\alpha(u) = -x$. Then $-u \leqslant^{\bowtie} -|z| \leqslant^{\bowtie} z$ and $y \leqslant^{\bowtie} |y| \leqslant^{\bowtie} |x| = -x \leqslant^{\bowtie} \alpha^{\bowtie}(u)$, so $x \leqslant^{\bowtie} \alpha^{\bowtie}(-u)$, $y \leqslant^{\bowtie} \alpha^{\bowtie}(u)$, and $(-u) \bullet u = -u \leqslant^{\bowtie} z$ gives the result.

<u>Subcase 2.2:</u> $-x \leqslant^{\bowtie} \alpha^{\bowtie}(|z|)$. Here we have $|y| \leqslant^{\bowtie} |x| = -x$ yields that $y \leqslant^{\bowtie} \alpha^{\bowtie}(|z|)$. Noting $z \bullet |z| = z \wedge |z| = z$, we have $x \leqslant^{\bowtie} \alpha^{\bowtie}(z)$, $y \leqslant^{\bowtie} \alpha^{\bowtie}(|z|)$, and $R_X(z, |z|, z)$. □

Lemma 7.3.7. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of SS. If $R_Y(\alpha^{\bowtie}(x), y, z)$, there exists $u, v \in X^{\bowtie}$ such that $R_X(x, u, v)$, $y \leqslant^{\bowtie} \alpha^{\bowtie}(u)$, and $\alpha^{\bowtie}(v) \leqslant^{\bowtie} z$.*

Proof. By the definition of R, $\alpha^{\bowtie}(z) \bullet y$ exists and $\alpha^{\bowtie}(x) \bullet y \leqslant^{\bowtie} z$. There are four cases, each with some subcases.

<u>**Case 1:**</u> $\alpha^{\bowtie}(x) \bullet y = \alpha^{\bowtie}(x) \vee y \leqslant^{\bowtie} z$. Here $\alpha^{\bowtie}(x) \leqslant^{\bowtie} z$ and $y \leqslant^{\bowtie} z$.

<u>**Subcase 1.1:**</u> $\alpha^{\bowtie}(x) \in Y$. From the fact that α is an Esakia map, there exists $u \in X$ with $x \leqslant u$ and $\alpha(u) = \alpha^{\bowtie}(u) = z$. Then $y \leqslant^{\bowtie} \alpha^{\bowtie}(u)$, $\alpha^{\bowtie}(u) \leqslant^{\bowtie} z$, and $R_X(x, u, u)$ since $x \bullet u \leqslant^{\bowtie} u$ is a consequence of $x \leqslant^{\bowtie} u$ by monotonicity and idempotence.

<u>**Subcase 1.2:**</u> $\alpha^{\bowtie}(x) \notin Y$. We may suppose $\alpha^{\bowtie}(x)$ and y are incomparable (i.e., since we are in the case where $\alpha^{\bowtie}(x) \bullet y = \alpha^{\bowtie}(x) \vee y$). Also, $-\alpha^{\bowtie}(x) = \alpha^{\bowtie}(-x) \in Y$ and $-z \leqslant^{\bowtie} \alpha^{\bowtie}(-x)$, $-z \leqslant^{\bowtie} -y$. Were it the case that $-z \in Y$, this would contradict the fact that Y is a forest. Hence $-z \notin Y$ and therefore $z \in Y$. The fact that $-z$ and $\alpha^{\bowtie}(-x)$ are comparable gives that z and $\alpha^{\bowtie}(-x)$ are comparable.

<u>**Subcase 1.2.1:**</u> $z \leqslant^{\bowtie} \alpha^{\bowtie}(-x)$. Here $y \leqslant^{\bowtie} \alpha^{\bowtie}(-x)$ and $\alpha^{\bowtie}(x) \leqslant^{\bowtie} -z \leqslant^{\bowtie} z$. We obtain the result from $-x \bullet x = x$, which gives $R_X(x, -x, x)$.

Subcase 1.2.2: $\alpha^{\bowtie}(-x) \leqslant^{\bowtie} z$. In this case, observe that $\alpha^{\bowtie}(x) \notin Y$ implies $\alpha^{\bowtie}(-x) \in Y$ and $-x \in X$. Then α being an Esakia function proves $u \in X$ with $-x \leqslant u$ and $\alpha(u) = \alpha^{\bowtie}(u) = z$. As $x \notin X$, we have $x \leqslant^{\bowtie} -x \leqslant^{\bowtie} u$ and this yields $x \bullet u \leqslant^{\bowtie} u$. Since $y \leqslant^{\bowtie} z = \alpha^{\bowtie}(u)$ and $\alpha^{\bowtie}(u) \leqslant^{\bowtie} z$ hold, we get the result from $R_X(x, u, u)$.

In all remaining cases, we may assume that $\alpha^{\bowtie}(x)$ and y are comparable and that not both of $\alpha^{\bowtie}(x) \in Y$ and $y \in Y$ hold.

Case 2: $|\alpha^{\bowtie}(x)| = |y|$. This gives $\alpha^{\bowtie}(x) \bullet y = \alpha^{\bowtie}(x) \wedge y$.

Subcase 2.1: $\alpha^{\bowtie}(x) \leqslant^{\bowtie} y$. Here we have $\alpha^{\bowtie}(x) \bullet y = \alpha^{\bowtie}(x) \leqslant^{\bowtie} z$. From $|\alpha^{\bowtie}(x)| = |y|$, we may obtain that $\alpha^{\bowtie}(x) = y$ or $\alpha^{\bowtie}(x) = -y$. If $\alpha^{\bowtie}(x) = y$, then $R_X(x, x, x)$ yields the result. If $\alpha^{\bowtie}(x) = -y$, then $\alpha^{\bowtie}(-x) = y$ and we use $R_X(x, -x, x)$ instead.

Subcase 2.2: $y \leqslant^{\bowtie} \alpha^{\bowtie}(x)$. In this setting $\alpha^{\bowtie}(x) \bullet y = y \leqslant^{\bowtie} z$. Again, $|\alpha^{\bowtie}(x)| = |y|$ provides that $\alpha^{\bowtie}(x) = y$ or $\alpha^{\bowtie}(x) = -y$. The former implies the result by noting that $R_X(x, x, x)$. The latter provides that $\alpha^{\bowtie}(-x) = y \leqslant^{\bowtie} z$, whence $R_X(x, -x, -x)$ proves the claim.

Case 3: $|y| < |\alpha^{\bowtie}(x)|$. Note that in this case $\alpha^{\bowtie}(x) \bullet y = \alpha^{\bowtie}(x) \leqslant^{\bowtie} z$.

Subcase 3.1: $y \leqslant^{\bowtie} \alpha^{\bowtie}(x)$. This subcase is immediate from $R_X(x, x, x)$.

Subcase 3.2: $\alpha^{\bowtie}(x) \leqslant^{\bowtie} y$. Here we may suppose $\alpha^{\bowtie}(x) \notin Y$, and therefore $\alpha^{\bowtie}(-x) \in Y$. This implies $\alpha^{\bowtie}(-x) = |\alpha^{\bowtie}(x)|$, whence $y \leqslant^{\bowtie} |y| \leqslant^{\bowtie} \alpha^{\bowtie}(-x)$. Then $R_X(x, -x, x)$ settles the third case.

Case 4: $|\alpha^{\bowtie}(x)| < |y|$. In this case we have $\alpha^{\bowtie}(x) \bullet y = y \leqslant^{\bowtie} z$.

Subcase 4.1: $\alpha^{\bowtie}(x), y \notin Y$. We have $|\alpha^{\bowtie}(x)| = -\alpha^{\bowtie}(x) \leqslant^{\bowtie} -y = |y|$. Hence $\alpha^{\bowtie}(-x) \leqslant -y$, and using the fact that α is an Esakia map gives $u \in Y$ with $-x \leqslant u$ and $\alpha^{\bowtie}(u) = \alpha(u) = -y$. It follows that $\alpha^{\bowtie}(-u) = y \leqslant^{\bowtie} z$. Thus $-u \leqslant^{\bowtie} x$, and from $-u, x \notin X$ we conclude that $x \bullet (-u) = -u$ since the value of $x \bullet (-u)$ is either the meet or the one with the larger absolute value. This implies $R_X(x, -u, -u)$ and $y = \alpha^{\bowtie}(-u) \leqslant^{\bowtie} z$ settles the subcase.

<u>Subcase 4.2:</u> $\alpha^{\bowtie}(x) \in Y$ and $y \notin Y$. Here $|\alpha^{\bowtie}(x)| = \alpha^{\bowtie}(x) \leqslant^{\bowtie} -y = |y|$. As α is an Esakia function, there exists $u \in X$ with $x \leqslant u$ and $\alpha^{\bowtie}(u) = \alpha(u) = -y$. Then $y = \alpha^{\bowtie}(-u)$ and $y \leqslant^{\bowtie} z$ hence yields $\alpha^{\bowtie}(-u) \leqslant^{\bowtie} z$. As $x \leqslant^{\bowtie} u$, $\bullet$ being monotone implies that $x \bullet (-u) \leqslant^{\bowtie} u \bullet (-u) = u \wedge -u \leqslant^{\bowtie} -u$. Hence $R_X(x, -u, -u)$, and since $y \leqslant^{\bowtie} \alpha^{\bowtie}(-u)$ and $\alpha^{\bowtie}(-u) \leqslant^{\bowtie} z$ this gives us the fourth case. $\square$

Lemma 7.3.8. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of SS. Then $\alpha^{\bowtie}$ is continuous.*

Proof. Let $U \cup V \subseteq \mathbf{Y}^{\bowtie}$ be open, where $U \subseteq Y$ and $V \subseteq D_Y^{\mathsf{c}}$ are open. The map $-\colon \mathbf{Y}^{\bowtie} \to \mathbf{Y}^{\bowtie}$ is a continuous bijection of compact Hausdorff spaces, whence it is a homeomorphism. Notice that $(\alpha^{\bowtie}))^{-1}[V]$ is precisely $\{x \in Y^{\bowtie} : -\alpha(-x) \in V\}$. This is the same as $\{-x \in Y^{\bowtie} : \alpha(-x) \in V\}$, so it is the inverse image of V under the continuous composite map $\alpha \circ -$. Thus the inverse image of V under this map is open. Because $(\alpha^{\bowtie})^{-1}[U \cup V] = (\alpha^{\bowtie})^{-1}[U] \cup (\alpha^{\bowtie})^{-1}[V]$, we obtain the lemma. $\square$

Lemma 7.3.9. *Let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of SS. Then $\alpha^{\bowtie}$ is a bounded morphism.*

Proof. This is immediate from the previous lemmas. $\square$

Lemma 7.3.10. $(-)_{\bowtie}\colon \mathsf{SM}_{\perp}^{\tau} \to \mathsf{SS}$ *is functorial.*

Proof. Let $\alpha\colon \mathbf{Y} \to \mathbf{Z}$ and $\beta\colon \mathbf{X} \to \mathbf{Y}$ be morphisms in $\mathsf{SM}_{\perp}^{\tau}$. We need

$$(\alpha \circ \beta)_{\bowtie} = \alpha_{\bowtie} \circ \beta_{\bowtie}.$$

Let $x \in X_{\bowtie}$. Then we have $(\alpha \circ \beta)_{\bowtie}(x) = \alpha(\beta(x)) = \alpha_{\bowtie}(\beta_{\bowtie}(x))$ as a consequence of the fact that $(-)_{\bowtie}$ acts by restriction. It is obvious that $(-)_{\bowtie}$ preserves the identity morphism. $\square$

Lemma 7.3.11. $(-)^{\bowtie}\colon \mathsf{SS} \to \mathsf{SM}_{\perp}^{\tau}$ *is functorial.*

Proof. Consider objects $\mathbf{X} = (X, \leqslant_{\mathbf{X}}, D_{\mathbf{X}}, \tau_{\mathbf{X}})$ and $\mathbf{Y} = (Y, \leqslant_{\mathbf{Y}}, D_{\mathbf{Y}}, \tau_{\mathbf{Y}})$, and $\mathbf{Z} = (Z, \leqslant_{\mathbf{Z}}, D_{\mathbf{Z}}, \tau_{\mathbf{Z}})$ of SS, and let $\alpha\colon \mathbf{Y} \to \mathbf{Z}$ and $\beta\colon \mathbf{X} \to \mathbf{Y}$ be morphisms of SS. Let $x \in X^{\bowtie}$. Either $x \in X$ or $x \in \{-y : y \notin D_{\mathbf{X}}\}$. In the first situation, we have

$$(\alpha \circ \beta)^{\bowtie}(x) = (\alpha \circ \beta)(x) = \alpha(\beta(x)) = \alpha^{\bowtie}(\beta^{\bowtie}(x)).$$

In the second situation, write $x = -y$ where $y \notin D_{\mathbf{X}}$. From this we get

$$(\alpha \circ \beta)^{\bowtie}(x) = -(\alpha \circ \beta)(y) = -\alpha(\beta(y)).$$

Also, $\beta^{\bowtie}(x) = -\beta(y)$ is not in Y, whence $\alpha^{\bowtie}(-\beta(y)) = -\alpha(\beta(y))$. Therefore $(\alpha \circ \psi)^{\bowtie} = \alpha^{\bowtie} \circ \psi^{\bowtie}$ in each case. It is obvious that $(-)^{\bowtie}$ preserves the identity morphism, so the lemma follows. $\square$

Lemma 7.3.12. *Let* $\mathbf{X} = (X, \leqslant, R, {}^{*}, I, \tau)$ *be an object of* $\mathsf{SM}_{\perp}^{\tau}$. *Then* $(\mathbf{X}_{\bowtie})^{\bowtie} \cong \mathbf{X}$.

Proof. Define $\theta_{\mathbf{X}}\colon (\mathbf{X}_{\bowtie})^{\bowtie} \to \mathbf{X}$ by

$$\theta_{\mathbf{X}}(x) = \begin{cases} x & \text{if } x \in I \\ (-x)^{*} & \text{if } x \notin I \end{cases}$$

This function is well-defined because $x \notin I$ implies that $-x \in I$ is an element of $\mathbf{X}$. We will prove that $\theta_{\mathbf{X}}$ is an isomorphism in $\mathsf{SM}_{\perp}^{\tau}$. It is enough to show that $\theta_{\mathbf{X}}$ is an order isomorphism, homeomorphism, preserves the involution, is an isomorphism with respect to R, and satisfies $\theta_{\mathbf{X}}[I] = I$.

We first show that $\theta_{\mathbf{X}}$ is an order isomorphism. Let $x, y \in (\mathbf{X}_{\bowtie})^{\bowtie}$ with $x \leqslant^{\bowtie} y$. If $x, y \in X_{\bowtie}$, then $\theta_{\mathbf{X}}(x) = x \leqslant y = \theta_{\mathbf{X}}(y)$. If $x, y \notin X_{\bowtie}$, then $-x, -y \in X_{\bowtie}$ and $x \leqslant^{\bowtie} y$ gives $-y \leqslant -x$, whence $(-x)^{*} \leqslant (-y)^{*}$. Then $\theta_{\mathbf{X}}(x) = (-x)^{*} \leqslant (-y)^{*} = \theta_{\mathbf{X}}(y)$.

If $x \notin X_{\bowtie}$ and $y \in X_{\bowtie}$, then $x \leqslant^{\bowtie} y$ gives that $-x$ and y are $\leqslant$-comparable. If $-x \leqslant y$, then $(-x)^* \leqslant -x \leqslant y$, and if $y \leqslant -x$, then $(-x)^* \leqslant y^* \leqslant y$. In both cases we obtain $\theta_{\mathbf{X}}(x) \leqslant \theta_{\mathbf{X}}(y)$. This shows that $\theta_{\mathbf{X}}$ is isotone.

To show $\theta_{\mathbf{X}}$ is order-reflecting, let $x, y \in (\mathbf{X}_{\bowtie})^{\bowtie}$ with $\theta_{\mathbf{X}}(x) \leqslant \theta_{\mathbf{X}}(y)$. If $x, y \in X_{\bowtie}$, then $x \leqslant^{\bowtie} y$ follows immediately. If $x, y \notin X_{\bowtie}$, then $(-x)^* \leqslant (-y)^*$, and thus $-y \leqslant -x$. In this case, $-x, -y \in X_{\bowtie}$, so $x \leqslant^{\bowtie} y$ by definition. If $x \in X_{\bowtie}$ and $y \notin X_{\bowtie}$, then $x = \theta_{\mathbf{X}}(x) \leqslant \theta_{\mathbf{X}}(y) = (-y)^*$. But $y \notin X_{\bowtie}$ gives $(-y)^* \notin X_{\bowtie}$, a contradiction to the fact that $X_{\bowtie}$ is an upset. In the last case, suppose that $x \notin X_{\bowtie}$ and $y \in X_{\bowtie}$. Then $(-x)^* \leqslant y$ by hypothesis. Since y and $-x$ are comparable, we get that $-x$ and y are comparable and that $-x, y \in X_{\bowtie}$. The definition of $\leqslant^{\bowtie}$ entails that $x = -(-x) \leqslant^{\bowtie} y$. $\theta_{\mathbf{X}}$ is thus order-reflecting.

To finish the proof that $\theta_{\mathbf{X}}$ is an order isomorphism, we must show surjectivity. Let $x \in X$. If $x \in I$, then $x \in (X_{\bowtie})^{\bowtie}$ and $\theta_{\mathbf{X}}(x) = x$. If $x \notin I$, then $x^* \in I$ and hence $-(x^*) \in (X_{\bowtie})^{\bowtie}$ and $-(x^*) \notin X_{\bowtie}$. Then $\theta_{\mathbf{X}}(-(x^*)) = (-(-(x^*)))^* = x^{**} = x$. Thus $\theta_{\mathbf{X}}$ is onto, whence it is an order isomorphism.

We next show that $\theta_{\mathbf{X}}$ is a homeomorphism. From the above, $\theta_{\mathbf{X}}$ is a bijection so from $(\mathbf{X}_{\bowtie})^{\bowtie}$ and $\mathbf{X}$ being compact Hausdorff spaces, it is enough to show that $\theta_{\mathbf{X}}$ is continuous. Let $W \subseteq X$ be open, and set $U = W \cap I$ and $V = W \cap I^{\mathsf{c}}$. As I is open by definition, both U and V are open as well. $\theta_{\mathbf{X}}^{-1}[U] = U$ by definition. Observe that $\theta_{\mathbf{X}}(x) \notin I$ implies that $x \notin I$ because $x \in I$ would imply $\theta_{\mathbf{X}}(x) = x$. From this, we have

$$\begin{aligned}\theta_{\mathbf{X}}^{-1}[V] &= \{x \in (X_{\bowtie})^{\bowtie} : \theta_{\mathbf{X}}(x) \in V\} \\ &= \{x \in (X_{\bowtie})^{\bowtie} : (-x)^* \in V\}\end{aligned}$$

Now $^*\colon \mathbf{X} \to \mathbf{X}$ and $-\colon (\mathbf{X}_{\bowtie})^{\bowtie} \to (\mathbf{X}_{\bowtie})^{\bowtie}$ are continuous bijections by definition, and the above is precisely the inverse image of V under the composition of $-$ and *. Thus V is an open subset of $(\mathbf{X}_{\bowtie})^{\bowtie}$ disjoint from $X_{\bowtie}$, and it follows that $\theta_{\mathbf{X}}^{-1}[W] = \theta_{\mathbf{X}}^{-1}[U] \cup \theta_{\mathbf{X}}^{-1}[V]$ is open. This gives that $\theta_{\mathbf{X}}$ is a homeomorphism.

To prove that $\theta_{\mathbf{X}}$ preserves the involution, let $x \in (\mathbf{X}_{\bowtie})^{\bowtie}$. If $-x \notin \mathbf{X}_{\bowtie}$, then $x \in \mathbf{X}_{\bowtie}$ and $\theta_{\mathbf{X}}(-x) = (-(-x))^* = x^* = \theta_{\mathbf{X}}(x)^*$. If $-x \in \mathbf{X}_{\bowtie}$ with $-x = x$, then $x = x^*$ and $\theta_{\mathbf{X}}(-x) = -x = x = x^* = \theta_{\mathbf{X}}(x)^*$. If $-x \in \mathbf{X}_{\bowtie}$ with $-x \neq x$, then $x \notin X_{\bowtie}$ and $\theta_{\mathbf{X}}(-x) = -x = (-x)^{**} = \theta_{\mathbf{X}}(x)^*$.

From the fact that $\theta_{\mathbf{X}}(x) = x$ for $x \in I$ we easily obtain $\theta_{\mathbf{X}}[I] = I$. All that is left is to prove that $\theta_{\mathbf{X}}$ is an isomorphism with respect to R. But this is an immediate consequence of the fact that R is determined by meet, join, and involution, and $\theta_{\mathbf{X}}$ is an involution-preserving order isomorphism. $\square$

Lemma 7.3.13. *Let $\mathbf{X}$ be an object of SS. Then $(\mathbf{X}^{\bowtie})_{\bowtie} \cong \mathbf{X}$.*

Proof. Let $i_{\mathbf{X}}\colon (\mathbf{X}^{\bowtie})_{\bowtie} \to \mathbf{X}$ be the identity map. Then $i_{\mathbf{X}}$ is an isomorphism of SS, and the result is immediate. $\square$

Theorem 7.3.14. *$(-)_{\bowtie}$ and $(-)^{\bowtie}$ give a covariant equivalence of categories between $\mathsf{SM}^{\tau}_{\perp}$ and SS.*

Proof. Naturality is all that remains to show. It is obvious that $i_{\mathbf{X}}$ gives a natural isomorphism. To prove this for $\theta_{\mathbf{X}}$, let $\alpha\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of $\mathsf{SM}^{\tau}_{\perp}$. It is enough to show that $\alpha \circ \theta_{\mathbf{X}} = \theta_{\mathbf{Y}} \circ (\alpha_{\bowtie})^{\bowtie}$, so let $x \in (X_{\bowtie})^{\bowtie}$. If $x \in X_{\bowtie}$, then taking x as the argument of the maps above gives $\alpha(x)$ on both sides of the equation. If $x \notin X_{\bowtie}$, then evaluating each side of the equation yields $\alpha(-x)^*$. This proves the claim. $\square$

Chapter 8

Dualized representations of srDL-algebras

The previous chapter provides a case study in how a duality-theoretic perspective can make an algebraic construction more transparent; we have seen that much of the complexity of the Galatos-Raftery construction dissolves when presented on dual spaces. In particular, the order-theoretic content of the construction is captured by simply reflecting points in the dual space across a designated subset, and the complicated multiplication inherent in the algebraic variant of the construction is captured dually by a simple piecewise-defined partial multiplication (compare: the multiplication in the algebraic variant of $(-)^{\bowtie}$ defined in Chapter 5, the partial multiplication dual variant of $(-)^{\bowtie}$ given in Chapter 7, and the definition of multiplication on $\mathbf{S}$ and $\mathbf{S}\backslash\{0\}$ in Example 2.3.8).

This chapter provides a second case study. Here we apply duality-theoretic methods to simplify the construction in [1] of srDL-algebras (see Section 2.3.1) from quadruples $(\mathbf{B}, \mathbf{A}, \vee_e, N)$, where $\mathbf{B}$ is a Boolean algebra, $\mathbf{A}$ is a GMTL-algebra, and $\vee_e$ and N are maps that parametrize how $\mathbf{B}$ and $\mathbf{A}$ are assembled. Our dual analogue of this construction builds the extended Priestley duals of srDL-algebras from the extended Priestley duals of $\mathbf{B}$ and $\mathbf{A}$, together with some data dualizing $\vee_e$ and N. The content of this chapter is based on the author's [27].

8.1 Algebraic representations by quadruples

We begin by recounting the pertinent aspects the Aguzzoli-Flaminio-Ugolini quadruples construction of [1].

Definition 8.1.1. *By an* algebraic quadruple *we mean an ordered tuple* $(\mathbf{B}, \mathbf{A}, \vee_e, N)$ *consisting of:*

- *A Boolean algebra* $\mathbf{B}$.
- *A GMTL-algebra* $\mathbf{A}$ *with* $B \cap A = \{1\}$.
- *A nucleus* $N \colon \mathbf{A} \to \mathbf{A}$ *that is also a lattice homomorphism (sometimes called a* wdl-admissible map*).*
- *An* external join $\vee_e$*, i.e., a map* $\vee_e \colon B \times A \to A$ *that satisfies the conditions enumerated below (where for each* $u \in B$ *and* $x \in A$*, we employ the abbreviations* $\nu_u(y) := u \vee_e y$ *and* $\lambda_x(v) := v \vee_e x$*),*

(V1) For every $u \in B$*, and* $x \in A$*,* ν_u *is an endomorphism of* $\mathbf{A}$ *and the map* λ_x *is a lattice homomorphism from (the lattice reduct of)* $\mathbf{B}$ *into (the lattice reduct of)* $\mathbf{A}$.

(V2) ν_0 *is the identity on* $\mathbf{A}$ *and* ν_1 *is constantly equal to* 1*, where* 0 *and* 1 *denote the bounds of* $\mathbf{B}$.

(V3) For all $u, v \in B$ *and for all* $x, y \in A$*,*

$$\nu_u(x) \vee \nu_v(y) = \nu_{u \vee v}(x \vee y) = \nu_u(\nu_v(x \vee y)).$$

If $(\mathbf{B}_1, \mathbf{A}_1, \vee_1, N_1)$ and $(\mathbf{B}_2, \mathbf{A}_2, \vee_2, N_2)$ are algebraic quadruples, say that a pair (h, k) is a *good morphism pair* provided it satisfies:

- $h\colon \mathbf{B}_1 \to \mathbf{B}_2$ is a homomorphism of Boolean algebras.
- $k\colon \mathbf{A}_1 \to \mathbf{A}_2$ is a homomorphism of GMTL-algebras.
- $k(u \vee_1 x) = h(u) \vee_2 k(x)$ whenever $(u, x) \in B \times A$.
- $k(N_1(x)) = N_2(k(x))$ for all $x \in A_1$.

With good morphisms pairs as arrows, algebraic quadruples form a category $\mathsf{Q_{GMTL}}$.

The construction that we aim to dualize proceeds as follows. Starting from an algebraic quadruple $(\mathbf{B}, \mathbf{A}, \vee_e, N)$, define a relation $\sim$ on on $B \times A$ by $(u, x) \sim (v, y)$ if and only if $u = v$, $\boldsymbol{\nu}_{\neg u}(x) = \boldsymbol{\nu}_{\neg u}(y)$, and $\boldsymbol{\nu}_u(N_{\mathbf{A}}(x)) = \boldsymbol{\nu}_u(N_{\mathbf{A}}(y))$. One may show that $\sim$ is an equivalence relation. We define an algebra

$$\mathbf{B} \otimes_e^N \mathbf{A} = (B \times A/{\sim}, \odot, \Rightarrow, \sqcap, \sqcup, [0, 1], [1, 1])$$

whose operations are defined on representatives $[u, x], [v, y] \in B \times A/{\sim}$ by

$$[u, x] \odot [v, y] = [u \wedge v, \boldsymbol{\nu}_{u \vee \neg v}(y \to x) \wedge \boldsymbol{\nu}_{\neg u \vee v}(x \to y) \wedge \boldsymbol{\nu}_{\neg u \vee \neg v}(x \cdot y)]$$

$$[u, x] \Rightarrow [v, y] = [u \to v, \boldsymbol{\nu}_{u \vee v}(N(y) \to N(x)) \wedge \boldsymbol{\nu}_{\neg u \vee v}(N(x \cdot y)) \wedge \boldsymbol{\nu}_{\neg u \vee \neg v}(x \to y)]$$

$$[u, x] \sqcap [v, y] = [u \wedge v, \boldsymbol{\nu}_{u \vee v}(x \vee y) \wedge \boldsymbol{\nu}_{u \vee \neg v}(x) \wedge \boldsymbol{\nu}_{\neg u \vee v}(y) \wedge \boldsymbol{\nu}_{\neg u \vee \neg v}(x \wedge y)]$$

$$[u, x] \sqcup [v, y] = [u \vee v, \boldsymbol{\nu}_{u \vee v}(x \wedge y) \wedge \boldsymbol{\nu}_{u \vee \neg v}(y) \wedge \boldsymbol{\nu}_{\neg u \vee v}(x) \wedge \boldsymbol{\nu}_{\neg u \vee \neg v}(x \vee y)]$$

It turns out that $\mathbf{B} \otimes_e^N \mathbf{A}$ is an srDL-algebra, and indeed $\otimes_e^N$ provides one functor of a categorical equivalence. In fact, for each subvariety H of GMTL, let $\mathsf{srDL_H}$ be the full subcategory of srDL whose objects are srDL-algebras $\mathbf{A}$ such that $\mathscr{R}(\mathbf{A}) \in \mathsf{H}$.[18]

[18] Recall that $\mathscr{R}(\mathbf{A})$ denotes the radical of $\mathbf{A}$. Radicals, coradicals, and Boolean skeletons of srDL-algebras are pervasive in this chapter. For pertinent definitions and basic results, see Section 2.3.1.

Moreover, let $\mathsf{Q_H}$ the full subcategory of $\mathsf{Q_{GMTL}}$ whose objects are algebraic quadruples $(\mathbf{B}, \mathbf{A}, \vee_e, N)$ such that $\mathbf{A} \in \mathsf{H}$. We may define functors $\Phi_{\mathsf{H}} : \mathsf{srDL_H} \to \mathsf{Q_{GMTL}}$ and $\Xi_{\mathsf{H}} : \mathsf{Q_{GMTL}} \to \mathsf{srDL_H}$ by

$$\begin{aligned}\Phi_{\mathsf{H}}(\mathbf{A}) &= (\mathscr{B}(\mathbf{A}), \mathscr{R}(\mathbf{A}), \vee, N_{\mathbf{A}})\\ \Phi_{\mathsf{H}}(k) &= (k{\restriction_{\mathscr{B}(\mathbf{A})}}, k{\restriction_{\mathscr{R}(\mathbf{A})}}),\end{aligned}$$

where $N_{\mathbf{A}} \colon \mathscr{R}(\mathbf{A}) \to \mathscr{R}(\mathbf{A})$ is the wdl-admissible map defined by $N_{\mathbf{A}}(x) = \neg\neg x$, and

$$\begin{aligned}\Xi_{\mathsf{H}}((\mathbf{B}, \mathbf{A}, \vee_e, N)) &= \mathbf{B} \otimes_e^N \mathbf{A}\\ \Xi_{\mathsf{H}}(h, k)([u, x]) &= [h(u), k(x)].\end{aligned}$$

From [1], $\mathsf{Q_H}$ and $\mathsf{srDL_H}$ are (covariantly) equivalent categories via the above functors.

Remark 8.1.2. A word on notation is in order. Because the construction outlined above involves many different types, we will make an effort to reserve a, b, c for general elements of srDL-algebras, whereas we will reserve u, v, w for Boolean elements and x, y, z for radical elements. Where possible, we will hold to the same convention for prime filters of these algebras, except that filters will be denoted by a Gothic typeface. Thus $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are used for prime filters of an srDL-algebra, whereas $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$ are used for ultrafilters of its Boolean skeleton and $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ are used for generalized prime filters of its radical.

8.2 Representing dual spaces by externally prime filter pairs

Our goal is to understand the extended Priestley dual of a given srDL-algebra in terms of the extended Priestley duals of its Boolean skeleton and radical, and we take our first steps in that direction in this section. For each srDL-algebra $\mathbf{A}$ and each $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$, an easy argument verifies that $\mathfrak{a} \cap \mathscr{B}(\mathbf{A})$ is an ultrafilter of $\mathscr{B}(\mathbf{A})$ and $\mathfrak{a} \cap \mathscr{R}(\mathbf{A})$ is a generalized prime filter of $\mathscr{R}(\mathbf{A})$.

Definition 8.2.1. *Let* $\mathbf{A} \in \mathsf{srDL}$. *Say that* $(\mathfrak{u}, \mathfrak{x}) \in \mathcal{S}(\mathscr{B}(\mathbf{A})) \times \mathcal{S}(\mathscr{R}(\mathbf{A}))$ *is* externally prime *if*

$$\forall (u, x) \in \mathscr{B}(\mathbf{A}) \times \mathscr{R}(\mathbf{A}), u \vee x \in \mathfrak{x} \textit{ implies } u \in \mathfrak{u} \textit{ or } x \in \mathfrak{x}. \tag{8.2.1}$$

Moreover, define

$$\mathcal{F}_{\mathbf{A}} = \{(\mathfrak{u}, \mathfrak{x}) \in \mathcal{S}(\mathscr{B}(\mathbf{A})) \times \mathcal{S}(\mathscr{R}(\mathbf{A})) : (\mathfrak{u}, \mathfrak{x}) \textit{ is externally prime}\}$$

Remark 8.2.2. We often understand $\mathcal{F}_{\mathbf{A}}$ as bearing the product order, i.e., we have $(\mathfrak{u}, \mathfrak{x}) \subseteq (\mathfrak{v}, \mathfrak{y})$ if and only if $\mathfrak{u} \subseteq \mathfrak{v}$ and $\mathfrak{x} \subseteq \mathfrak{y}$. Because $\mathfrak{u}$ and $\mathfrak{v}$ are ultrafilters (and in particular maximal), the condition that $\mathfrak{u} \subseteq \mathfrak{v}$ is equivalent to $\mathfrak{u} = \mathfrak{v}$.

The definition of the functor $\Phi_{\mathsf{H}}(\mathbf{A})$ employs the wdl-admissible map $N_{\mathbf{A}}$ on $\mathscr{R}(\mathbf{A})$ defined by $N_{\mathbf{A}}(x) = \neg\neg x$, and this nucleus will be fundamental to our investigation. As for any nucleus, $N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]$ is a residuated lattice in its own right, and we observe that for each prime filter $\mathfrak{x}$ of $N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]$ we have

$$N_{\mathbf{A}}^{-1}[\mathfrak{x}] = \max\{\mathfrak{y} \in \mathcal{S}(\mathscr{R}(\mathbf{A})) : \mathrm{N}_{\mathbf{A}}[\mathfrak{y}] = \mathfrak{x}\}.$$

It is simple to verify this by checking that $N_{\mathbf{A}}^{-1}[\mathfrak{x}]$ is a prime filter, and that any $\mathfrak{y}$ with $N_{\mathbf{A}}[\mathfrak{y}] = \mathfrak{x}$ is contained in $N_{\mathbf{A}}^{-1}[\mathfrak{x}]$.

In order to tie the duals of srDL-algebras to the duals of their radicals and Boolean skeletons, we will give a representation of $\mathcal{S}(\mathbf{A})$, $\mathbf{A} \in \mathsf{srDL}$, in terms of externally prime filter pairs. However, it turns out that only some of the points in $\mathcal{S}(\mathbf{A})$ may be represented by such filter pairs (in fact, the members of $\mathcal{F}_{\mathbf{A}}$ turn out to correspond to those prime filters of $\mathbf{A}$ that do not contain all of $\mathscr{R}(\mathbf{A})$, as we shall see). In order to represent every $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$, we create a (modified) copy of some points and place them "above" the poset $\mathcal{F}_{\mathbf{A}}$ (cf. the dual construction of $(-)^{\bowtie}$ by a reflection "below" the set of designated elements). To achieve this, define

$$\mathcal{F}_{\mathbf{A}}^{\partial} = \{+(\mathfrak{u}, \mathfrak{y}) : (\mathfrak{u}, \mathfrak{y}) \in P\}$$

where

$$P = \{(\mathfrak{u}, \mathfrak{y}) \in \mathcal{S}(\mathscr{B}(\mathbf{A})) \times \mathcal{S}(N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]) : (\mathfrak{u}, N_{\mathbf{A}}^{-1}[\mathfrak{y}]) \in \mathcal{F}_{\mathbf{A}} \text{ and } N_{\mathbf{A}}^{-1}[\mathfrak{y}] \neq \mathscr{R}(\mathbf{A})\}.$$

The decoration + comes by analogy from our work in Chapter 7, and intuitively we think of $\mathcal{F}_{\mathbf{A}}^{\partial}$ as corresponding to an intuitively "upper" or "positive" piece of $\mathcal{S}(\mathbf{A})$. The following definition makes this precise.

Definition 8.2.3. *Let $\mathcal{F}_{\mathbf{A}}^{\bowtie} := \mathcal{F}_{\mathbf{A}} \dot\cup \mathcal{F}_{\mathbf{A}}^{\partial}$, and define a partial order $\sqsubseteq$ on $\mathcal{F}_{\mathbf{A}}^{\bowtie}$ by $\mathfrak{p} \sqsubseteq \mathfrak{q}$ if and only if one of the following holds.*

1. *$\mathfrak{p} = (\mathfrak{u}, \mathfrak{x})$ and $\mathfrak{q} = (\mathfrak{v}, \mathfrak{y})$ for some $(\mathfrak{u}, \mathfrak{x}), (\mathfrak{v}, \mathfrak{y}) \in \mathcal{F}_{\mathbf{A}}$ with $(\mathfrak{u}, \mathfrak{x}) \subseteq (\mathfrak{v}, \mathfrak{y})$.*
2. *$\mathfrak{p} = +(\mathfrak{u}, \mathfrak{x})$ and $\mathfrak{q} = +(\mathfrak{v}, \mathfrak{y})$ for some $+(\mathfrak{u}, \mathfrak{x}), +(\mathfrak{v}, \mathfrak{y}) \in \mathcal{F}_{\mathbf{A}}^{\partial}$ with $(\mathfrak{v}, \mathfrak{y}) \subseteq (\mathfrak{u}, \mathfrak{x})$.*
3. *$\mathfrak{p} = (\mathfrak{u}, \mathfrak{x})$ and $\mathfrak{q} = +(\mathfrak{v}, \mathfrak{y})$ for some $(\mathfrak{u}, \mathfrak{x}) \in \mathcal{F}_{\mathbf{A}}$, $(\mathfrak{v}, \mathfrak{y}) \in \mathcal{F}_{\mathbf{A}}^{\partial}$ with $\mathfrak{u} = \mathfrak{v}$.*

Our definition of the pair $(\mathfrak{u}, \mathfrak{x})$ being externally prime seems to intrinsically depend on $\mathfrak{u}$ and $\mathfrak{x}$ being filters (i.e., as opposed to abstract points in some Priestley space). However, we presently provide an entirely abstract treatment of external primality, for which the following observation is crucial.

Let $\mathfrak{a} \in \mathcal{S}(A)$, where $\mathbf{A} \in \mathsf{srDL}$. For each $u \in \mathscr{B}(\mathbf{A})$ we have $u \vee \neg u = 1 \in \mathfrak{a}$ by Lemma 2.3.6(2), and since $\mathfrak{a}$ is prime one of $u \in \mathfrak{a}$ or $\neg u \in \mathfrak{a}$ must hold. This implies that each $\mathfrak{a} \in \mathcal{S}(A)$ contains an ultrafilter of $\mathscr{B}(\mathbf{A})$. Because ultrafilters are maximal and each $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$ is proper, this ultrafilter is unique.

Definition 8.2.4. *For $\mathfrak{a} \in \mathcal{S}(A)$ denote by $\mathfrak{u}_{\mathfrak{a}}$ the unique ultrafilter $\mathfrak{u}$ of $\mathscr{B}(\mathbf{A})$ with $\mathfrak{u} \subseteq \mathfrak{a}$. We call $\mathfrak{u}_{\mathfrak{a}}$ the* ultrafilter of $\mathfrak{a}$.

We say that an ultrafilter $\mathfrak{u} \subseteq \mathscr{B}(\mathbf{A})$ fixes *$\mathfrak{x} \in \mathcal{S}(\mathscr{R}(A))$ if there exists $\mathfrak{a} \in \mathcal{S}(A)$ with $\mathfrak{u} \subseteq \mathfrak{a}$ (equivalently $\mathfrak{u} = \mathfrak{u}_{\mathfrak{a}}$) and $\mathfrak{x} = \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$.*

It is obvious that $\mathfrak{u}_{\mathfrak{a}}$ fixes $\mathfrak{a} \cap \mathscr{R}(\mathbf{A})$ for each $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$. In order to explain the terminology of an ultrafilter "fixing" a radical filter,[19] we define for each $u \in \mathscr{B}(\mathbf{A})$ a map $\mu_u \colon \mathcal{S}(\mathscr{R}(\mathbf{A})) \to \mathcal{S}(\mathscr{R}(\mathbf{A}))$ by

$$\mu_u(\mathfrak{x}) = \{x \in \mathscr{R}(\mathbf{A}) : u \vee x \in \mathfrak{x}\} = \nu_u^{-1}[\mathfrak{x}],$$

where the notation $\nu_u(x) = u \vee x$ was introduced in Definition 8.1.1. Observe that μ_u is the extended Priestley dual of the GMTL-endomorphism ν_u. The following technical lemma gives some useful properties of the maps μ_u.

Lemma 8.2.5. *Let $\mathbf{A} \in \mathsf{srDL}$, let $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$, and let $u, v \in \mathscr{B}(\mathbf{A})$. Then the following hold.*

1. *$\mu_{u \vee v}(\mathfrak{x})$ is one of $\mu_u(\mathfrak{x})$ or $\mu_v(\mathfrak{x})$.*

[19] Note that we call the elements of $\mathcal{S}(\mathscr{R}(\mathbf{A}))$ *radical filters*.

2. $\mu_{u\wedge v}(\mathfrak{x})$ *is one of* $\mu_u(\mathfrak{x})$ *or* $\mu_v(\mathfrak{x})$.

3. $\mu_u(\mathfrak{x}) = \mathfrak{x}$ *or* $\mu_{\neg u}(\mathfrak{x}) = \mathfrak{x}$.

4. $\mu_u(\mathfrak{x}) = \mathfrak{x}$ *or* $\mu_u(\mathfrak{x}) = \mathscr{R}(\mathbf{A})$.

5. $\mu_u(\mu_u(\mathfrak{x})) = \mu_u(\mathfrak{x})$.

Proof. Let $x \in \mathscr{R}(\mathbf{A})$. If either of $x \vee u \in \mathfrak{x}$ or $x \vee v \in \mathfrak{x}$ holds, then $x \vee u \vee v \in \mathfrak{x}$ as $\mathfrak{x}$ is an up-set. Also, $x \vee (u \wedge v) \in \mathfrak{x}$ gives $x \vee u \in \mathfrak{x}$ and $x \vee v \in \mathfrak{x}$. These facts provide that $\mu_u(\mathfrak{x}), \mu_v(\mathfrak{x}) \subseteq \mu_{u\vee v}(\mathfrak{x})$ and $\mu_{u\wedge v}(\mathfrak{x}) \subseteq \mu_u(\mathfrak{x}), \mu_v(\mathfrak{x})$.

For (1), suppose on the contrary that both of $\mu_u(\mathfrak{x}) \subset \mu_{u\vee v}(\mathfrak{x})$ and $\mu_v(\mathfrak{x}) \subset \mu_{u\vee v}(\mathfrak{x})$. It follows that there exist $x, y \in \mathscr{R}(\mathbf{A})$ with $x \vee u \vee v, y \vee u \vee v \in \mathfrak{x}$, but $x \vee u \notin \mathfrak{x}$ and $x \vee v \notin \mathfrak{x}$. From $\mathfrak{x}$ being an up-set and $x \vee u \vee v, y \vee u \vee v \in \mathfrak{x}$, this means $x \vee y \vee u \vee v \in \mathfrak{x}$. As $\mathfrak{x}$ is prime in $\mathscr{R}(\mathbf{A})$ and $x \vee u, y \vee v \in \mathscr{R}(\mathbf{A})$, it follows that $(x \vee u) \vee (y \vee v) = x \vee y \vee u \vee v \in \mathfrak{x}$ implies $x \vee u \in \mathfrak{x}$ or $y \vee v \in \mathfrak{x}$. This is a contradiction, so either $\mu_{u\vee v}(\mathfrak{x}) = \mu_u(\mathfrak{x})$ or $\mu_{u\vee v}(\mathfrak{x}) = \mu_v(\mathfrak{x})$.

For (2), suppose on the contrary that $\mu_{u\wedge v}(\mathfrak{x}) \subset \mu_u(\mathfrak{x})$ and $\mu_{u\wedge v}(\mathfrak{x}) \subset \mu_v(\mathfrak{x})$. Then there are $x, y \in \mathscr{R}(\mathbf{A})$ with $x \vee (u \wedge v), y \vee (u \wedge v) \notin \mathfrak{x}$ but $x \vee u \in \mathfrak{x}$ and $y \vee v \in \mathfrak{x}$. Distributivity of the lattice reduct implies that $(x \vee u) \wedge (x \vee v) \notin \mathfrak{x}$, and as $x \vee u \in \mathfrak{x}$ we have $x \vee v \notin \mathfrak{x}$. Likewise, $(y \vee u) \wedge (y \vee v) \notin \mathfrak{x}$ and $y \vee v \in \mathfrak{x}$ together imply that $y \vee u \notin \mathfrak{x}$. Since $\mathfrak{x}$ is prime, this yields $x \vee y \vee u \vee v \notin \mathfrak{x}$. This contradicts $x \vee u \in \mathfrak{x}$ since $x \vee u \leqslant x \vee y \vee u \vee v$ and $\mathfrak{x}$ is is an up-set, giving (2).

For (3), we have $\mathfrak{x} = \mu_0(\mathfrak{x}) = \mu_{u\wedge\neg u}(\mathfrak{x})$, which is either $\mu_u(\mathfrak{x})$ or $\mu_{\neg u}(\mathfrak{x})$ from (2).

For (4), suppose that $\mu_u(\mathfrak{x}) \neq \mathfrak{x}$ and $\mathfrak{x} \neq \mathscr{R}(\mathbf{A})$ (so in particular $\mu_u(\mathfrak{x}) \neq \mathscr{R}(\mathbf{A})$). Item (3) gives that $\mu_{\neg u}(\mathfrak{x}) = \mathfrak{x}$, and $\mathscr{R}(\mathbf{A}) = \mu_1(\mathfrak{x}) = \mu_{u\vee\neg u}(\mathfrak{x})$ gives $\mu_u(\mathfrak{x}) = \mathscr{R}(\mathbf{A})$ or $\mu_{\neg u}(\mathfrak{x}) = \mathscr{R}(\mathbf{A})$ from (1). Since $\mathfrak{x} \neq \mathscr{R}(\mathbf{A})$, the second of these possibilities is excluded. Thus $\mathscr{R}(\mathbf{A}) = \mu_u(\mathfrak{x})$.

Item (5) is a direct consequence of (4). $\square$

Lemma 8.2.6. *Let* $\mathbf{A}$ *be an srDL-algebra. Then we have the following.*

1. $\mathfrak{a} \cap \mathscr{R}(\mathbf{A})$ *is a generalized prime filter of* $\mathscr{R}(\mathbf{A})$*, and is a fixed-point of each of the maps* μ_u *for* $u \notin \mathfrak{u}_{\mathfrak{a}}$.

2. *Conversely, if* $\mathfrak{x} = \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$ *is proper and* $\mathfrak{u}$ *is an ultrafilter of* $\mathscr{B}(\mathbf{A})$ *such that* $\mathfrak{x}$ *is fixed by each* μ_u *for* $u \notin \mathfrak{u}$*, then* $\mathfrak{u} \subseteq \mathfrak{a}$*. In particular,* $\mathfrak{u} = \mathfrak{u}_{\mathfrak{a}}$.

Proof. To prove (1), note first that $\mathfrak{a} \cap \mathscr{R}(\mathbf{A}) \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$ is an obvious consequence of the definitions. To prove the rest, let $u \notin \mathfrak{u}_{\mathfrak{a}}$ and set $\mathfrak{x} = \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$. For $x \in \mathfrak{x}$ we have that $x \leqslant u \vee x$ implies $u \vee x \in \mathfrak{x}$, whence $x \in \mu_u(\mathfrak{x})$. Thus $\mathfrak{x} \subseteq \mu_u(\mathfrak{x})$. For the reverse inclusion, let $x \in \mu_u(\mathfrak{x})$. Then $u \vee x \in \mathfrak{x}$, and as $\mathfrak{x} \subseteq \mathfrak{a}$ we get $u \vee x \in \mathfrak{a}$. Since $\mathfrak{a}$ is a prime filter of $\mathbf{A}$, it follows that $u \in \mathfrak{a}$ or $x \in \mathfrak{a}$. As $u \notin \mathfrak{u}_{\mathfrak{a}}$, the first of these cannot occur. Thus $x \in \mathfrak{a}$. It follows that $x \in \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$, giving $\mu_u(\mathfrak{x}) = \mathfrak{x}$ as claimed.

To prove (2), let $u \in \mathfrak{u}$. Then $\mathfrak{u}$ being an ultrafilter implies $\neg u \notin \mathfrak{u}$, so $\mathfrak{x}$ is a fixed-point of $\mu_{\neg u}$ by assumption. Were $u \notin \mathfrak{a}$, we would have $\neg u \in \mathfrak{a}$ since $\mathfrak{a}$ is prime and $u \vee \neg u \in \mathfrak{a}$. Let $x \in \mathscr{R}(\mathbf{A})$. Then $\neg u, x \leqslant \neg u \vee x$, and as both $\mathscr{R}(\mathbf{A})$ and $\mathfrak{a}$ are up-sets it follows that $\neg u \vee x \in \mathfrak{a} \cap \mathscr{R}(\mathbf{A}) = \mathfrak{x}$. Since $\mathfrak{x}$ is fixed by $\mu_{\neg u}$, this implies that $x \in \mu_{\neg u}(\mathfrak{x}) = \mathfrak{x}$. Therefore $\mathscr{R}(\mathbf{A}) \subseteq \mathfrak{x}$, a contradiction to the assumption that $\mathfrak{x}$ is proper. We thus obtain $u \in \mathfrak{a}$, and $\mathfrak{u} \subseteq \mathfrak{a}$. Since the ultrafilter of $\mathfrak{a}$ is unique by the remarks above, we get $\mathfrak{u} = \mathfrak{u}_{\mathfrak{a}}$ as well. $\square$

Remark 8.2.7. Observe that it would be more natural to work with prime ideals of $\mathscr{B}(\mathbf{A})$ rather than ultrafilters. By the above, $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$ fixes a proper filter $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$ if and only if $\mu_u(\mathfrak{x}) = \mathfrak{x}$ for $u \in \mathfrak{u}^{\mathsf{c}}$, and the sets of the form $\mathfrak{u}^{\mathsf{c}}$ for $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$ are exactly the prime ideals of $\mathscr{B}(\mathbf{A})$. Because we have adopted a variant of Priestley duality that employs prime filters rather than prime ideals, we will continue working with filters in the present setting.

Lemma 8.2.8. *Let $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$. Then there is $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$ such that $\mathfrak{u}$ fixes $\mathfrak{x}$.*

Proof. We will use the prime ideal theorem for distributive lattices. For this, let $\mathfrak{i} = \mathscr{R}(\mathbf{A})\backslash\mathfrak{x}$ and observe that $\mathfrak{i}$ is an ideal of $\mathscr{R}(\mathbf{A})$. Also, its down-set

$$\downarrow\mathfrak{i} = \{a \in A : a \leqslant i \text{ for some } i \in \mathfrak{i}\}$$

is an ideal of $\mathbf{A}$ as well. Note moreover that $\mathfrak{x}$ is a filter of $\mathbf{A}$ (and not just of $\mathscr{R}(\mathbf{A})$). Since $\mathfrak{x} \cap \downarrow\mathfrak{i} = \varnothing$, there exists $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$ such that $\mathfrak{a} \cap \downarrow\mathfrak{i} = \varnothing$ and $\mathfrak{x} \subseteq \mathfrak{a}$. It is easy to see that $\mathfrak{u}_{\mathfrak{a}}$ fixes $\mathfrak{x}$, which settles the claim. □

Remark 8.2.9. Note that Lemma 8.2.8 also shows that an arbitrary $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$ is of the form $\mathfrak{x} = \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$ for some $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$.

Every radical filter is fixed by at least one ultrafilter by the foregoing lemma. One consequence of the following is that a given radical filter may be fixed by many ultrafilters.

Lemma 8.2.10. *Let $\mathbf{A} \in \mathsf{srDL}$ and $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$. Then $\mathfrak{u}$ fixes $\mathscr{R}(\mathbf{A})$.*

Proof. We must show that there exists $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$ such that $\mathfrak{a} \cap \mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{A})$ and $\mathfrak{u} \subseteq \mathfrak{a}$. Let $\mathfrak{f}$ be the filter of $\mathbf{A}$ generated by $\mathfrak{u} \cup \mathscr{R}(\mathbf{A})$. Then $\mathfrak{f}$ is proper. To see this, toward a contradiction suppose that $0 \in \mathfrak{f}$. Then there exists $u \in \mathfrak{u}$ and $x \in \mathscr{R}(\mathbf{A})$ such that $u \wedge x \leqslant 0$. From the fact that $a \cdot b \leqslant a \wedge b$ holds in every integral CRL, we get $u \cdot x \leqslant 0$. Thus $x \leqslant u \to 0 = \neg u$ by residuating. As $\mathscr{R}(\mathbf{A})$ is an up-set, this implies $\neg u \in \mathscr{R}(\mathbf{A})$. The only Boolean element in $\mathscr{R}(\mathbf{A})$ is 1 (see, e.g., [1]), whence $\neg u = 1$. From this we obtain $u = 0$, a contradiction to the assumption that $\mathfrak{u}$ is an ultrafilter (i.e., since ultrafilters are proper). Thus $\mathfrak{f} \neq A$, and we may extend $\mathfrak{f}$ to a prime filter $\mathfrak{a}$ of $\mathbf{A}$ by the prime filter theorem. It is easy to see that $\mathfrak{u} \subseteq \mathfrak{a}$ and $\mathscr{R}(\mathbf{A}) \subseteq \mathfrak{a}$, which proves the lemma. □

The following lemma provides a crucial step in making external primality extrinsic (i.e., rendering external primality on abstract spaces rather than spaces whose points are filters).

Lemma 8.2.11. *Let* $\mathbf{A} \in \mathsf{srDL}$ *and let* $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$. *Then* $(\mathfrak{u}, \mathfrak{x})$ *is externally prime if and only if* $\mathfrak{u}$ *fixes* $\mathfrak{x}$.

Proof. For the forward implication, let $(\mathfrak{u}, \mathfrak{x}) \in \mathcal{F}_{\mathbf{A}}$. If $\mathfrak{x} = \mathscr{R}(\mathbf{A})$, then the result follows from Lemma 8.2.10. Suppose $\mathfrak{x} \neq \mathscr{R}(\mathbf{A})$. We will apply Lemma 8.2.6, so let $u \notin \mathfrak{u}$. Since $\mathfrak{x} \subseteq \mu_u(\mathfrak{x})$ always holds, it is enough to show that $\mu_u(\mathfrak{x}) \subseteq \mathfrak{x}$ and we let $x \in \mu_u(\mathfrak{x})$. Then $u \vee x \in \mathfrak{x}$, so by external primality we have that $u \in \mathfrak{u}$ or $x \in \mathfrak{x}$. Since $u \notin \mathfrak{u}$ by assumption, it follows that $x \in \mathfrak{x}$. Thus $\mu_u(\mathfrak{x}) = \mathfrak{x}$ for every $u \notin \mathfrak{u}$, and the result follows from Lemma 8.2.6(2).

For the backward implication, suppose that $\mathfrak{u}$ fixes $\mathfrak{x}$. Let $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$ be such that $u \vee x \in \mathfrak{x}$, and suppose that $u \notin \mathfrak{u}$. Then from Lemma 8.2.6(1) and $\mathfrak{u}$ fixing $\mathfrak{x}$, we get that $\mathfrak{x} = \mu_u(\mathfrak{x})$. But $x \in \mu_u(\mathfrak{x})$ means $u \vee x \in \mathfrak{x}$, whence $x \in \mathfrak{x}$. This implies that $u \in \mathfrak{u}$ or $x \in \mathfrak{x}$, so $(\mathfrak{u}, \mathfrak{x})$ is externally prime. □

The next two lemmas are never invoked in the sequel, but provide some intuition about ultrafilters that fix a given radical filter. For each radical filter $\mathfrak{x}$, define

$$\mathfrak{f}_{\mathfrak{x}} = \bigcap \{\mathfrak{u} : \mathfrak{u} \text{ fixes } \mathfrak{x}\}$$

Notice that $\mathfrak{f}_{\mathfrak{x}}$ is a nonempty and proper filter, and $\mathfrak{f}_{\mathfrak{x}}$ is an ultrafilter if and only if there is just one ultrafilter fixing $\mathfrak{x}$.

Lemma 8.2.12. *Let* $u \notin \mathfrak{f}_{\mathfrak{x}}$. *Then* μ_u *fixes* $\mathfrak{x}$.

Proof. Note that if $u \notin \mathfrak{f}_{\mathfrak{x}}$ then there exists an ultrafilter $\mathfrak{u}$ of $\mathscr{B}(\mathbf{A})$ with $\mathfrak{u}$ fixing $\mathfrak{x}$ and $u \notin \mathfrak{u}$. Then Lemma 8.2.6 provides $\mu_u(\mathfrak{x}) = \mathfrak{x}$. □

The filter $\mathfrak{f}_{\mathfrak{x}}$ characterizes exactly which ultrafilters fix $\mathfrak{x}$.

Lemma 8.2.13. *Let $\mathfrak{u}$ be an ultrafilter of $\mathscr{B}(\mathbf{A})$ and let $\mathfrak{x} \in \mathcal{S}(A)$. Then $\mathfrak{u}$ fixes $\mathfrak{x}$ if and only if $\mathfrak{f}_{\mathfrak{x}} \subseteq \mathfrak{u}$.*

Proof. It is obvious that if $\mathfrak{u}$ fixes $\mathfrak{x}$, then $\mathfrak{f}_{\mathfrak{x}} \subseteq \mathfrak{u}$. For the converse, observe that if $\mathfrak{x} = \mathscr{R}(\mathbf{A})$, then Lemma 8.2.10 implies that $\mathfrak{u}$ fixes $\mathfrak{x}$. If $\mathfrak{x} \neq \mathscr{R}(\mathbf{A})$, then by Remark 8.2.9 we have that there is $\mathfrak{a} \in \mathcal{S}(A)$ with $\mathfrak{x} = \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$. Let $u \notin \mathfrak{u}$. Note that $u \notin \mathfrak{f}_{\mathfrak{x}}$ since $\mathfrak{f}_{\mathfrak{x}} \subseteq \mathfrak{u}$, and from Lemma 8.2.12 we get μ_u fixes $\mathfrak{x}$. It follows that $\mathfrak{x}$ is fixed by each map μ_u for $u \notin \mathfrak{u}$, and by Lemma 8.2.6(2) this yields that $\mathfrak{u} \subseteq \mathfrak{a}$. It follows that $\mathfrak{u}$ fixes $\mathfrak{x}$, settling the claim. $\square$

The following technical lemma helps to link external primality of filter pairs to prime filters of srDL-algebras.

Proposition 8.2.14. *Let $\mathbf{A} \in \mathsf{srDL}$, let $(\mathfrak{u}, \mathfrak{x}) \in \mathcal{S}(\mathscr{B}(\mathbf{A})) \times \mathcal{S}(\mathscr{R}(\mathbf{A}))$, and let*

$$\mathfrak{p} = \langle \mathfrak{u} \cup \mathfrak{x} \rangle$$

be the filter of $\mathbf{A}$ generated by $\mathfrak{u} \cup \mathfrak{x}$. Then if $(\mathfrak{u}, \mathfrak{x})$ is externally prime (equivalently, if $\mathfrak{u}$ fixes $\mathfrak{x}$), we have that $\mathfrak{p}$ is prime.

Proof. The filters $\mathfrak{u}$ and $\mathfrak{x}$ being closed under $\wedge$ implies that

$$\mathfrak{p} = \{a \in A \mid u \wedge x \leqslant a \text{ for some } u \in \mathfrak{u}, x \in \mathfrak{x}\}.$$

In order to prove $\mathfrak{p}$ is prime, we will make use of the decomposition of elements in an srDL-algebra in terms of Boolean and radical elements (see Equation 2.3.1 of Section 2.3.1). Let $a_1 \vee a_2 \in \mathfrak{p}$, and write

$$a_1 = (u_1 \vee \neg x_1) \wedge (\neg u_1 \vee x_1)$$

$$a_2 = (u_2 \vee \neg x_2) \wedge (\neg u_2 \vee x_2)$$

for some $u_1, u_2 \in \mathscr{B}(\mathbf{A})$ and $x_1, x_2 \in \mathscr{R}(\mathbf{A})$. We must prove that $a_1 \in \mathfrak{p}$ or $a_2 \in \mathfrak{p}$, and that $\mathfrak{p} \neq A$. Since $a_1 \vee a_2 \in \mathfrak{p}$, there exist $u \in \mathfrak{u}$ and $x \in \mathfrak{x}$ such that $u \wedge x \leqslant a_1 \vee a_2$. A calculation using the distributivity of the lattice reduct of $\mathbf{A}$ shows that

$$\begin{aligned} a_1 \vee a_2 = ((u_1 \vee u_2) \vee (\neg x_1 \vee \neg x_2)) \wedge ((u_1 \vee \neg u_2) \vee x_2) \wedge ((\neg u_1 \vee u_2) \vee x_1) \\ \wedge((\neg u_1 \vee \neg u_2) \vee (x_1 \vee x_2)). \end{aligned}$$

The right-hand side of the above is a meet, and this implies that $u \wedge x$ is a lower bound of each of the meetands $(u_1 \vee u_2) \vee (\neg x_1 \vee \neg x_2)$, $(u_1 \vee \neg u_2) \vee x_2$, $(\neg u_1 \vee u_2) \vee x_1$, and $(\neg u_1 \vee \neg u_2) \vee (x_1 \vee x_2)$.

We further scrutinize the first of these, viz. $u \wedge x \leqslant (u_1 \vee u_2) \vee (\neg x_1 \vee \neg x_2)$. This inequality holds if and only if $u \leqslant u_1 \vee u_2$. In order to prove this, recall that $\mathbf{A}$ is isomorphic to $\mathscr{B}(\mathbf{A}) \otimes_e^{N_{\mathbf{A}}} \mathscr{R}(\mathbf{A})$ via the construction of Section 8.1, and in particular there are isomorphisms

$$\lambda_B \colon \mathscr{B}(\mathbf{A}) \to \mathscr{B}(\mathscr{B}(\mathbf{A}) \otimes_e^{N_{\mathbf{A}}} \mathscr{R}(\mathbf{A}))$$

$$\lambda_R \colon \mathscr{R}(\mathbf{A}) \to \mathscr{R}(\mathscr{B}(\mathbf{A}) \otimes_e^{N_{\mathbf{A}}} \mathscr{R}(\mathbf{A})).$$

By direct computation, we obtain:

$$\begin{aligned} (\lambda_B(b) \sqcap \lambda_R(x)) &= [u, 1] \sqcap [1, x] = [u, \neg u \vee x], \\ \lambda_B(u_1 \vee u_2) \sqcup \neg\lambda_R(x_1 \wedge x_2) &= [u_1 \vee u_2, 1] \sqcup [0, x_1 \wedge x_2] \\ &= [u_1 \vee u_2, (u_1 \vee u_2) \vee (x_1 \wedge x_2)]. \end{aligned}$$

Further, $[u, \neg u \vee x] \wedge [u_1 \vee u_2, (u_1 \vee u_2) \vee (x_1 \wedge x_2)] = [u \wedge (u_1 \vee u_2), \bar{x}]$, where $\bar{x} \in \mathscr{R}(\mathbf{A})$ is some term of the radical calculated via the operations given Section 8.1. Using the isomorphism we may obtain that $u \wedge x \leqslant (u_1 \vee u_2) \vee (\neg x_1 \vee \neg x_2)$ holds if and only if $[u \wedge (u_1 \vee u_2), \bar{x}] = [u, \neg u \vee x]$, and this holds, which in turn holds if and only if $u \leqslant u_1 \vee u_2$. Since $\mathfrak{u}$ is prime in $\mathscr{B}(\mathbf{A})$, it follows that not both of $u_1 \notin \mathfrak{u}$, $u_2 \notin \mathfrak{u}$ may hold. Now note that each of

$$(u_1 \vee \neg u_2) \vee x_2$$

$$(\neg u_1 \vee u_2) \vee x_1$$

$$(\neg u_1 \vee \neg u_2) \vee (x_1 \vee x_2)$$

is in $\mathscr{R}(\mathbf{A})$ since the latter is an up-set. Observe that if $y \in \mathscr{R}(\mathbf{A})$ with $u \wedge x \leqslant y$, then we have $x \leqslant \neg u \vee y$ by residuating and applying Lemma 2.3.6(4). This gives $\neg u \vee y \in \mathfrak{x}$, and from external primality we get that either $\neg u \in \mathfrak{u}$ or $y \in \mathfrak{x}$. But $\neg u \in \mathfrak{u}$ is impossible since $u \in \mathfrak{u}$, whence $y \in \mathfrak{x}$. We may apply this argument to the three terms above to obtain the following conclusions:

$$x_2 \in \mathfrak{x} \text{ or } u_1 \vee \neg u_2 \in \mathfrak{u}$$

$$x_1 \in \mathfrak{x} \text{ or } \neg u_1 \vee u_2 \in \mathfrak{u}$$

$$x_1 \vee x_2 \in \mathfrak{x} \text{ or } \neg u_1 \vee \neg u_2 \in \mathfrak{u}.$$

From the above, we have:

$$u_1, \neg u_2 \in \mathfrak{u} \implies a_1 \in \mathfrak{p}$$

$$\neg u_1, u_2 \in \mathfrak{u} \implies a_2 \in \mathfrak{p}$$

$$u_1, u_2 \in \mathfrak{u} \text{ and } x_1 \in \mathfrak{x} \implies a_1 \in \mathfrak{p}$$

$$u_1, u_2 \in \mathfrak{u} \text{ and } x_2 \in \mathfrak{x} \implies a_2 \in \mathfrak{p}.$$

Since not both of $u_1 \notin \mathfrak{u}$, $u_2 \notin \mathfrak{u}$ hold and $x_1 \in \mathfrak{x}$ or $x_2 \in \mathfrak{x}$, it follows that $a_1 \in \mathfrak{p}$ or $a_2 \in \mathfrak{p}$. To finish the proof, note that $\mathfrak{p}$ is proper if $\mathfrak{u}$ is proper: $u \wedge x > 0$ for any $u \in \mathfrak{u}$ and $x \in \mathfrak{x}$, whence $0 \notin \mathfrak{p}$. It is immediate that $\mathfrak{p} \cap \mathscr{B}(\mathbf{A}) = \mathfrak{u}$, and $\mathfrak{p} \cap \mathscr{R}(\mathbf{A}) = \mathfrak{x}$ as $u \wedge x \leqslant a$ for $a \in \mathscr{R}(\mathbf{A})$ implies $a \in \mathfrak{x}$ by the above. □

The following indicates an especially important application of Proposition 8.2.14.

Definition 8.2.15. *Let $\mathbf{A} \in \mathsf{srDL}$ and let $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$. Define $R_{\mathfrak{u}} := \langle \mathfrak{u} \cup \mathscr{R}(\mathbf{A}) \rangle$.*

Note that $R_{\mathfrak{u}} \in \mathcal{S}(\mathbf{A})$ follows immediately from Proposition 8.2.14 and Lemma 8.2.10.

Lemma 8.2.16. *Let $\mathbf{A} \in \mathsf{srDL}$. The following hold.*

1. *If $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$, $N_{\mathbf{A}}[\mathfrak{x}] \neq N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]$, and $\mathfrak{u}$ fixes $\mathfrak{x}$, then we have*

$$\langle \mathfrak{u} \cup \mathfrak{x} \rangle^* = \{a \in A \mid u \wedge \neg x \leqslant a, \text{ for some } u \in \mathfrak{u}, \neg\neg x \in N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})] \backslash N_{\mathbf{A}}[\mathfrak{x}]\} \tag{8.2.2}$$

2. *Under the hypotheses of (1),*

$$\langle \mathfrak{u} \cup \mathfrak{x} \rangle^* \cap \mathscr{C}(\mathbf{A}) = \{\neg x : \neg\neg x \in N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})] \backslash N_{\mathbf{A}}[\mathfrak{x}]\}$$

   and $\langle \mathfrak{u} \cup \mathfrak{x} \rangle \subseteq \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$.

3. *If $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$ with $N_{\mathbf{A}}[\mathfrak{x}] = N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]$, then $\langle \mathfrak{u} \cup \mathfrak{x} \rangle^* = R_{\mathfrak{u}}$.*

Proof. To prove item (1), we check Equation 8.2.2 directly. Let $a \in A$ be such that $u \wedge \neg x \leqslant a$ for some $u \in \mathfrak{u}, \neg\neg x \in N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})] \backslash N_{\mathbf{A}}[\mathfrak{x}]$. Then $\neg a \leqslant \neg u \vee \neg\neg x$.

Were it the case that $\neg a \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle$, we would have $\neg u \vee \neg\neg x \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle$. But this contradicts $u \in \mathfrak{u}, \neg x \notin \mathfrak{u}, \neg\neg x \notin N_{\mathbf{A}}[\mathfrak{x}] \subseteq \mathfrak{x}$, whence we have $\neg a \notin \langle \mathfrak{u} \cup \mathfrak{x} \rangle$. It follows that $a \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$. This proves that the right-hand side of Equation 8.2.2 is contained in $\langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$

For the reverse inclusion, let $a \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$. Then by definition $\neg a \notin \langle \mathfrak{u} \cup \mathfrak{x} \rangle$. We again invoke Equation 2.3.1 of Section 2.3.1, and write $a = (u \wedge x) \vee (\neg u \wedge \neg x)$ for some Boolean element u and radical element x. It follows from this decomposition that $\neg a = (\neg u \wedge \neg\neg x) \vee (u \wedge \neg x)$ by the representation given in Section 8.1. Note that if $u \in \mathfrak{u}$, then from Lemma 2.3.7(2) we have $a \geqslant u \wedge x \geqslant u \wedge \neg y$ for every $y \in \mathscr{R}(\mathbf{A})$. Note that there exists $z \in \mathscr{R}(\mathbf{A})$ such that $\neg\neg z \notin \mathfrak{x}$ and $a \geqslant u \wedge \neg z$ since $N_{\mathbf{A}}[\mathfrak{x}] \neq N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]$. To see why, observe that if otherwise, $\neg u \in \mathfrak{u}$ provided that $u \notin \mathfrak{u}$, and since $\neg u \wedge \neg\neg x \leqslant \neg a \notin \langle \mathfrak{u} \cup \mathfrak{x} \rangle$, we get that $\neg\neg x \notin \mathfrak{x}$. It follows that $a \geqslant \neg u \wedge \neg x$ and $\neg\neg x \in N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})] \backslash N_{\mathbf{A}}[\mathfrak{x}]$. (1) follows.

For (2), we first show $\langle \mathfrak{u} \cup \mathfrak{x} \rangle \subseteq \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$. Let $a \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle$, so that $u \wedge x \leqslant a$ for some $u \in \mathfrak{u}$ and $x \in \mathfrak{x}$. As above, we have $\neg\neg x \in N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})] \backslash N_{\mathbf{A}}[\mathfrak{x}]$ and $u \wedge \neg x \leqslant u \wedge x \leqslant a$, whence $a \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$. For the rest of (2), note that

$$\langle \mathfrak{u} \cup \mathfrak{x} \rangle^* \cap \mathscr{C}(\mathbf{A}) = \{\neg x : \neg\neg x \in N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})] \backslash N_{\mathbf{A}}[\mathfrak{x}]\}$$

follows directly from the definition of *.

For (3), note that $\langle \mathfrak{u} \cup \mathfrak{x} \rangle^* \subseteq R_{\mathfrak{u}}$ follows from another computation using Equation 2.3.1 of Section 2.3.1. To prove the reverse inclusion, let $a \in R_{\mathfrak{u}}$ and let $u \in \mathfrak{u}$ and $x \in \mathfrak{x}$ be such that $u \wedge x \leqslant a$. It follows that $\neg a \leqslant \neg u \vee \neg x$. Note that if $\neg a \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle$, then by primality $\neg u \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle$ or $\neg x \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle$. From Lemma 2.3.7 we get that $\neg u \notin \mathfrak{u}$ and $\neg x \in \mathscr{C}(\mathbf{A})$, whence $u, \neg x \notin \langle \mathfrak{u} \cup \mathfrak{x} \rangle$. This implies $\neg a \notin \langle \mathfrak{u} \cup \mathfrak{x} \rangle$, and thus $a \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$. $\square$

Although srDL-algebras are not involutive, the negation operation $\neg$ greatly influences their structure. In the next several lemmas, we identify pertinent properties of the Routley star $*$ on the dual spaces of srDL-algebras.

Lemma 8.2.17. *Let* $\mathbf{A} \in \mathsf{srDL}$ *and let* $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$*. Then either* $\mathfrak{a} \subseteq \mathfrak{a}^*$ *or* $\mathfrak{a}^* \subseteq \mathfrak{a}$*.*

Proof. Suppose that $\mathfrak{a} \nsubseteq \mathfrak{a}^*$, and let $a \in \mathfrak{a}$ with $a \notin \mathfrak{a}^*$. Then $\neg a \in \mathfrak{a}$, whence $a, \neg a \in \mathfrak{a}$ and $a \wedge \neg a \in \mathfrak{a}$. Since srDL-algebras have normal i-lattice reducts, $a \wedge \neg a \leqslant b \vee \neg b$ holds for any elements a and b. For $b \in \mathfrak{a}^*$, we thus get $b \vee \neg b \in \mathfrak{a}$ as filters are up-sets. From the primality of $\mathfrak{a}$, we get $b \in \mathfrak{a}$ or $\neg b \in \mathfrak{a}$. In the latter situation, we would have $b \notin \mathfrak{a}^*$, contradicting the fact that b was chosen from $\mathfrak{a}^*$. Thus $b \in \mathfrak{a}$, so $\mathfrak{a}^* \subseteq \mathfrak{a}$. $\square$

Observe that if $\mathfrak{a} \subseteq \mathfrak{b}$ for some $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}(A)$, then it immediately follows that $\mathfrak{u}_{\mathfrak{a}} = \mathfrak{u}_{\mathfrak{b}}$. This implicates the following definition.

Definition 8.2.18. *Let* $\mathbf{A} \in \mathsf{srDL}$ *and let* $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$*. Define*

$$\mathcal{S}_{\mathfrak{u}} := \{\mathfrak{a} \in \mathcal{S}(A) : \mathfrak{u} = \mathfrak{u}_{\mathfrak{a}}\} = \{\mathfrak{a} \in \mathcal{S}(A) : \mathfrak{u} \subseteq \mathfrak{a}\}.$$

We call $\mathcal{S}_{\mathfrak{u}}$ *the* site of $\mathfrak{u}$ in $\mathbf{A}$*.*

Lemma 8.2.19. *Let* $\mathbf{A} \in \mathsf{srDL}$ *and let* $\mathfrak{a} \in \mathcal{S}(A)$*. Then* $\mathfrak{a}$ *and* $\mathfrak{a}^*$ *have the same ultrafilter. Consequently,* $\mathcal{S}_{\mathfrak{u}}$ *is closed under* $*$ *for every* $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$*.*

Proof. This is immediate from Lemma 8.2.17 and the remarks above. $\square$

Lemma 8.2.20. *Let* $\mathbf{A} \in \mathsf{srDL}$ *and let* $\mathfrak{a} \in \mathcal{S}(A)$*. Then one of* $\mathfrak{a}$ *or* $\mathfrak{a}^*$ *contains* $\mathscr{R}(\mathbf{A})$*.*

Proof. Let $a \in \mathscr{R}(\mathbf{A}) \backslash \mathfrak{a}$. Note that Lemma 2.3.7 gives $\neg a < a$ for each $a \in \mathscr{R}(\mathbf{A})$, whence $\neg a \notin \mathfrak{a}$. This follows because if $\neg a \in \mathfrak{a}$ were to hold, then $a \in \mathfrak{a}$ as $\mathfrak{a}$ is an up-set. From this we obtain $a \in \mathfrak{a}^*$ and thus $\mathscr{R}(\mathbf{A}) \backslash \mathfrak{a} \subseteq \mathfrak{a}^*$.

Assume that $\mathscr{R}(\mathbf{A}) \nsubseteq \mathfrak{a}$. Then we have $\mathscr{R}(\mathbf{A})\backslash\mathfrak{a} \neq \varnothing$, and the previous paragraph implies that there is $a \in \mathfrak{a}^*$ with $a \notin \mathfrak{a}$. As $\mathfrak{a} \subseteq \mathfrak{a}^*$ or $\mathfrak{a}^* \subseteq \mathfrak{a}$ by Lemma 8.2.17, it follows that $\mathfrak{a} \subseteq \mathfrak{a}^*$. Therefore $\mathscr{R}(\mathbf{A})\backslash\mathfrak{a} \subseteq \mathfrak{a}^*$ and $\mathscr{R}(\mathbf{A}) \cap \mathfrak{a} \subseteq \mathfrak{a} \subseteq \mathfrak{a}^*$, whence $\mathscr{R}(\mathbf{A}) = (\mathscr{R}(\mathbf{A}) \cap \mathfrak{a}) \cup (\mathscr{R}(\mathbf{A})\backslash\mathfrak{a}) \subseteq \mathfrak{a}^*$. □

Lemma 8.2.21. *Let* $\mathbf{A} \in \mathsf{srDL}$ *and let* $\mathfrak{a} \in \mathcal{S}(A)$*. Then either* $\mathfrak{a} \subseteq \mathscr{R}_{\mathfrak{u}_\mathfrak{a}} \subseteq \mathfrak{a}^*$ *or* $\mathfrak{a}^* \subseteq \mathscr{R}_{\mathfrak{u}_\mathfrak{a}} \subseteq \mathfrak{a}$*.*

Proof. Note that one of $\mathfrak{a} \subseteq \mathfrak{a}^*$ or $\mathfrak{a}^* \subseteq \mathfrak{a}$ holds by Lemma 8.2.17. Suppose that $\mathfrak{a}^* \subseteq \mathfrak{a}$, and set $\mathfrak{u} := \mathfrak{u}_\mathfrak{a}$. Note that $\mathscr{R}(\mathbf{A}) \subseteq \mathfrak{a}$ from Lemma 8.2.20, and this yields that $R_\mathfrak{u} = \langle \mathscr{R}(\mathbf{A}) \cup \mathfrak{u} \rangle \subseteq \mathfrak{a}$. We consider two cases.

Case 1: $R_\mathfrak{u} \nsubseteq \mathfrak{a}^*$. We will show that $\mathfrak{a}^* \subseteq R_\mathfrak{u}$. Let $a \in \mathfrak{a}^*$, and using Equation 2.3.1 write $a = (u \wedge x) \vee (\neg u \wedge \neg x)$ for some $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$. As $\mathfrak{a}^*$ is prime, one of $u \wedge x \in \mathfrak{a}^*$ or $\neg u \wedge \neg x \in \mathfrak{a}^*$ holds. Were it the case that $u \notin \mathfrak{u}$, this would imply that $\neg u \wedge \neg x \in \mathfrak{a}^*$ and $\neg x \in \mathfrak{a}^*$. But $\mathfrak{a}^*$ being an up-set and $\neg x \leqslant y$ for every $y \in \mathscr{R}(\mathbf{A})$ together imply that $\mathscr{R}(\mathbf{A}) \subseteq \mathfrak{a}^*$, a contradiction to the assumption. This entails that $u \in \mathfrak{u}$ and $u \wedge x \in \mathfrak{a}^*$. Because $u \wedge x \leqslant a$ and $u \wedge x \in R_\mathfrak{u}$, it follows that $\mathfrak{a}^* \subseteq R_\mathfrak{u}$.

Case 2: $R_\mathfrak{u} \subseteq \mathfrak{a}^*$. Pick $x \in \mathscr{C}(\mathbf{A})$. Lemma 2.3.7 entails that

$$\neg x \in \mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}_\mathfrak{u} \subseteq \mathfrak{a}^* \subseteq \mathfrak{a}.$$

Then as $x \leqslant \neg\neg x$ we get that $x \notin \mathfrak{a}^*$ and $\neg\neg x \notin \mathfrak{a}$ by the definition of $*$. It follows that $x \notin \mathfrak{a}$. Now let $a \in \mathfrak{a}$, and applying Equation 2.3.1 again write $a = (u \wedge y) \vee (\neg u \wedge \neg y)$ for some $u \in \mathscr{B}(\mathbf{A})$ and $y \in \mathscr{R}(\mathbf{A})$. We must have either $u \wedge y \in \mathfrak{a}$ or $\neg u \wedge \neg y \in \mathfrak{a}$ by primality. The comments above imply that since $\neg y \in \mathscr{C}(\mathbf{A})$, we have $\neg y \notin \mathfrak{a}$. Hence $u \wedge y \in \mathfrak{a}$, which implies $u \in \mathfrak{u}$ and $u \wedge y \in \mathscr{R}_\mathfrak{u}$. This shows that $\mathscr{R}_\mathfrak{u} = \mathfrak{a} = \mathfrak{a}^*$ as $u \wedge y \leqslant a$.

If instead $\mathfrak{a} \subseteq \mathfrak{a}^*$, then $\mathfrak{a} \subseteq R_{\mathfrak{u}_\mathfrak{a}} \subseteq \mathfrak{a}^*$ follows by similar reasoning. $\square$

Lemma 8.2.22. *Let* $\mathbf{A} \in \mathsf{srDL}$ *and let* $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$. *Then* $R_\mathfrak{u} = R_\mathfrak{u}^*$.

Proof. Let $a \in R_\mathfrak{u}$, and from Equation 2.3.1 write

$$a = (\neg u \vee x) \wedge (u \vee \neg x) = (u \wedge x) \vee (\neg u \wedge \neg x)$$

for some $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$. Observe that in every srDL-algebra, we have that $u \wedge y \leqslant \neg x$ iff $u = 0$ for every $u \in \mathscr{B}(\mathbf{A}), x, y \in \mathscr{R}(\mathbf{A})$; this may be shown in $\mathscr{B}(\mathbf{A}) \otimes_e^{N_\mathbf{A}} \mathscr{R}(\mathbf{A})$ and using the fact that Boolean elements, radical elements, and coradical elements have the form $[u, 1], [1, x]$, and $[0, y]$, respectively (see [1] for details). It follows that $\neg x \notin R_\mathfrak{u}$, whence $u \wedge x \in R_\mathfrak{u}$ and $u \in \mathfrak{u}$. Note that $\neg a = (u \wedge \neg x) \vee (\neg u \wedge \neg\neg x)$, and suppose that $\neg a \in R_\mathfrak{u}$. It follows that either $u \wedge \neg x \in R_\mathfrak{u}$ or $\neg u \wedge \neg\neg x \in R_\mathfrak{u}$. But $u \wedge \neg x \in R_\mathfrak{u}$ implies $\neg x \in R_\mathfrak{u}$ and $\neg u \wedge \neg\neg x \in R_\mathfrak{u}$ implies $\neg u \in R_\mathfrak{u}$, and each of these is a contradiction. Therefore $\neg a \notin R_\mathfrak{u}$, whence $a \in R_\mathfrak{u}^*$ and $R_\mathfrak{u} \subseteq R_\mathfrak{u}^*$.

For the reverse inclusion, let $a \in R_\mathfrak{u}^*$. Then $\neg a \notin R_\mathfrak{u}$ and by Equation 2.3.1, Section 2.3.1, there exist $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$ with

$$a = (u \wedge x) \vee (\neg u \wedge \neg x)$$

$$\neg a = (u \wedge \neg x) \vee (\neg u \wedge \neg\neg x).$$

Notice that if $u \notin \mathfrak{u}$, we have $\neg u \in \mathfrak{u}$. This would imply $\neg u \wedge \neg\neg x \in R_\mathfrak{u}$ since $\neg\neg x \in \mathscr{R}(\mathbf{A})$, entailing that $\neg a \in R_\mathfrak{u}$ as $\neg u \wedge \neg\neg x \leqslant \neg a$. This is a contradiction, so $u \in \mathfrak{u}$. It follows that $u \wedge x \in R_\mathfrak{u}$, whence $a \in R_\mathfrak{u}$ and $R_\mathfrak{u} = R_\mathfrak{u}^*$. $\square$

For each $\mathbf{A} \in \mathsf{srDL}$, we define a map $\alpha_{\mathbf{A}} \colon \mathcal{S}(\mathbf{A}) \to \mathcal{F}_{\mathbf{A}}^{\bowtie}$ by

$$\alpha_{\mathbf{A}}(\mathfrak{a}) = \begin{cases} (\mathfrak{a} \cap \mathscr{B}(\mathbf{A}),\, \mathfrak{a} \cap \mathscr{R}(\mathbf{A})), & \text{if } \mathfrak{a} \subseteq \mathfrak{a}^*, \\ +(\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}), N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]) & \text{otherwise,} \end{cases}$$

Note that by Lemma 8.2.17, the second clause obtains precisely when $\mathfrak{a}^* \subset \mathfrak{a}$. Since $\mathbf{A}$ is usually clear from context, we will typically write $\alpha_{\mathbf{A}}$ simply as α.

Lemma 8.2.23. *Let $\mathbf{A} \in \mathsf{srDL}$. Then $\alpha_{\mathbf{A}}$ is well-defined.*

Proof. All that demands verification is that the output of $\alpha_{\mathbf{A}}$ is in $\mathcal{F}_{\mathbf{A}}^{\bowtie}$. Let $\mathfrak{a} \in \mathcal{S}(\mathbf{A})$. Suppose first that $\mathfrak{a} \subseteq \mathfrak{a}^*$, and let $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$ with $u \vee x \in \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$. From $\mathfrak{a}$ being prime we have that $u \in \mathfrak{a}$ (in which case $u \in \mathfrak{a} \cap \mathscr{B}(\mathbf{A})$) or $x \in \mathfrak{a}$ (in which case $x \in \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$). This gives that $\alpha_{\mathbf{A}}(\mathfrak{a}) \in \mathcal{F}_{\mathbf{A}}^{\bowtie}$.

Now suppose that if $\mathfrak{a}^* \subset \mathfrak{a}$. It is straightforward to verify that $N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]$ is a prime filter of $N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]$, and we need only check that

$$(\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}), N_{\mathbf{A}}^{-1}[N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]]) \in \mathcal{F}_{\mathbf{A}}.$$

Let $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$ with $u \vee x \in N_{\mathbf{A}}^{-1}[N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]]$. Then

$$N_{\mathbf{A}}(u \vee x) = \neg\neg(u \vee x) = u \vee \neg\neg x \in N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})] \subseteq \mathfrak{a}^* \cap \mathscr{R}(\mathbf{A}) \subseteq \mathfrak{a}^*.$$

Since $\mathfrak{a}^*$ is a prime filter, it follows that $u \in \mathfrak{a}^*$ or $N_{\mathbf{A}}(x) \in \mathfrak{a}^*$. From this we have $u \in \mathfrak{a}^* \cap \mathscr{B}(\mathbf{A})$ or else $N_{\mathbf{A}}(x) \in N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]$. Since the latter implies $x \in N_{\mathbf{A}}^{-1}[N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]]$, the result follows. □

Lemma 8.2.24. *Let $\mathbf{A} \in \mathsf{srDL}$. Then $\alpha_{\mathbf{A}}$ is a bijection.*

Proof. One may readily show the injectivity of $\alpha_{\mathbf{A}}$ by using the representation offered in Equation 2.3.1. We address surjectivity, so first let $(\mathfrak{u}, \mathfrak{x}) \in \mathcal{F}_{\mathbf{A}}$. Set

$\mathfrak{a} := \langle \mathfrak{u} \cup \mathfrak{x} \rangle$, and observe that Proposition 8.2.14 gives that $\mathfrak{a} \in \mathcal{S}(A)$. From Lemma 8.2.16 we have also that $\mathfrak{a} \cap \mathscr{B}(\mathbf{A}) = \mathfrak{u}$, that $\mathfrak{a} \cap \mathscr{R}(\mathbf{A}) = \mathfrak{x}$, and that $\mathfrak{a} \subseteq \mathfrak{a}^*$. It follows from this that $\alpha_{\mathbf{A}}(\mathfrak{a}) = (\mathfrak{u}, \mathfrak{x})$.

Second, let $+(\mathfrak{u}, \mathfrak{y}) \in \mathcal{F}^{\partial}_{\mathbf{A}}$. Then if we set $\mathfrak{x} := N^{-1}_{\mathbf{A}}[\mathfrak{y}]$, we have $(\mathfrak{u}, \mathfrak{x}) \in \mathcal{F}_{\mathbf{A}}$. Let $\mathfrak{a} = \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$. Then $\mathfrak{a}^* \subseteq \mathfrak{a}$ from Lemma 8.2.16, and $\mathfrak{a} \cap \mathscr{B}(\mathbf{A}) = \mathfrak{u}$ from from Lemma 8.2.19. Direct computation shows that $N_{\mathbf{A}}[\mathfrak{b}^* \cap \mathscr{R}(\mathbf{A})] = \neg(\mathscr{C}(\mathbf{A}) \backslash \mathfrak{b})$ for each $\mathfrak{b} \in \mathcal{S}(A)$. It follows from Lemma 8.2.16 that

$$\mathscr{C}(\mathbf{A}) \backslash \mathfrak{a} = \{\neg x : \neg\neg x \in N_{\mathbf{A}}[\mathfrak{y}] \cap N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]\},$$

whence $\neg(\mathscr{C}(\mathbf{A}) \backslash \mathfrak{a}) = N_{\mathbf{A}}[\mathfrak{y}]$. This yields $\alpha_{\mathbf{A}}(\mathfrak{a}) = +(\mathfrak{u}, \mathfrak{y})$, giving surjectivity. $\square$

Theorem 8.2.25. *Let* $\mathbf{A}$ *be a srDL-algebra. Then* $\mathcal{S}(\mathbf{A})$ *and* $\mathcal{F}^{\bowtie}_{\mathbf{A}}$ *are order-isomorphic.*

Proof. We show that $\alpha_{\mathbf{A}}$ is an order isomorphism, for which it suffices (by Lemma 8.2.24) to show that if $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{S}(A)$, then

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \text{ iff } \alpha_{\mathbf{A}}(\mathfrak{a}_1) \leqslant \alpha_{\mathbf{A}}(\mathfrak{a}_2)$$

It is easy to see from the definition that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ implies $\alpha_{\mathbf{A}}(\mathfrak{a}_1) \leqslant \alpha_{\mathbf{A}}(\mathfrak{a}_2)$, so we address the converse. Suppose that $\alpha_{\mathbf{A}}(\mathfrak{a}_1) \leqslant \alpha_{\mathbf{A}}(\mathfrak{a}_2)$, and abbreviate

$$\mathfrak{u}_1 := \mathfrak{a}_1 \cap \mathscr{B}(\mathbf{A})$$

$$\mathfrak{x}_1 := \mathfrak{a}_1 \cap \mathscr{R}(\mathbf{A})$$

$$\mathfrak{y}_1 := N_{\mathbf{A}}[\mathfrak{a}_1 \cap \mathscr{R}(\mathbf{A})]$$

$$\mathfrak{u}_2 := \mathfrak{a}_2 \cap \mathscr{B}(\mathbf{A})$$

$$\mathfrak{x}_2 := \mathfrak{a}_2 \cap \mathscr{R}(\mathbf{A})$$

$$\mathfrak{y}_2 := N_{\mathbf{A}}[\mathfrak{a}_2 \cap \mathscr{R}(\mathbf{A})]$$

We consider four cases.

<u>**Case 1:**</u> $\mathfrak{a}_1 \subseteq \mathfrak{a}_1^*$ and $\mathfrak{a}_2 \subseteq \mathfrak{a}_2^*$. Note that in this case, by hypothesis we have $\mathfrak{a}_1 \cap \mathscr{R}(\mathbf{A}) \subseteq \mathfrak{a}_2 \cap \mathscr{R}(\mathbf{A})$ and $\mathfrak{a}_1 \cap \mathscr{B}(\mathbf{A}) = \mathfrak{a}_2 \cap \mathscr{B}(\mathbf{A})$. Let $a \in \mathfrak{a}_1$, and by Equation 2.3.1 let $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$ be such that $a = (\neg u \vee x) \wedge (u \vee \neg x)$. Then $\neg u \vee x, u \vee \neg x \in \mathfrak{a}_1$. By primality and $\neg u \vee x \in \mathfrak{a}_1$, we get that $\neg u \in \mathfrak{u}_1 \subseteq \mathfrak{u}_2$ or $x \in \mathfrak{x}_1 \subseteq \mathfrak{x}_2$. In either case, $\neg u \vee x \in \mathfrak{a}_2$. Since $u \vee \neg x \in \mathfrak{a}_1$ we get that $u \in \mathfrak{a}_1$ (as a consequence of $\neg x \notin \mathfrak{a}_1$ by Lemma 8.2.21 and the fact that $\neg x \leqslant y$ for every $y \in \mathscr{R}(\mathbf{A})$), we have that $u \vee \neg x \in \mathfrak{a}_2$. It follows from this that $a \in \mathfrak{a}_2$, giving $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$.

<u>**Case 2:**</u> $\mathfrak{a}_1^* \subseteq \mathfrak{a}_1$ and $\mathfrak{a}_2 \subseteq \mathfrak{a}_2^*$. This case is impossible from the definition of the order on $\mathcal{F}_{\mathbf{A}}^{\bowtie}$.

<u>**Case 3:**</u> $\mathfrak{a}_1 \subseteq \mathfrak{a}_1^*$ and $\mathfrak{a}_2^* \subseteq \mathfrak{a}_2$. The hypothesis implies that $\mathfrak{u}_1 = \mathfrak{u}_2$, whence by Lemma 8.2.21 we have $\mathfrak{a}_1 \subseteq R_{\mathfrak{u}_1} \subseteq \mathfrak{a}_2$.

<u>**Case 4:**</u> $\mathfrak{a}_1^* \subseteq \mathfrak{a}_1$ and $\mathfrak{a}_2^* \subseteq \mathfrak{a}_2$. Because $N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})] = \neg(\mathscr{C}(\mathbf{A})\backslash\mathfrak{a})$ for each $\mathfrak{a} \in \mathcal{S}(A)$, it follows that $\neg(\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_2) \subseteq \neg(\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_1)$. From this we may obtain that $\mathfrak{a}_1 \cap \mathscr{C}(\mathbf{A}) \subseteq \mathfrak{a}_2 \cap \mathscr{C}(\mathbf{A})$. To see this, note that $\neg(\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_2) \subseteq \neg(\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_1)$ yields $\neg\neg(\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_2) \subseteq \neg\neg(\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_1)$, and as $\neg\neg(\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_i) = (\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_i)$ (for $i = 1, 2$), we have that $\mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_2 \subseteq \mathscr{C}(\mathbf{A})\backslash\mathfrak{a}_1$. Hence $\mathfrak{a}_1 \cap \mathscr{C}(\mathbf{A}) \subseteq \mathfrak{a}_2 \cap \mathscr{C}(\mathbf{A})$. Let $a \in \mathfrak{a}_1$, and as usual we write $a = (\neg u \vee x) \wedge (u \vee \neg x)$ for some $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$. Then $\neg u \vee x, u \vee \neg x \in \mathfrak{a}_1$ since $\mathfrak{a}_1$ is an up-set. As an arbitrary $x \in \mathscr{R}(\mathbf{A})$ is both in $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in the present case, we get $\neg u \vee x$ is in $\mathfrak{a}_2$. Since $u \vee \neg x \in \mathfrak{a}_1$, primality gives $u \in \mathfrak{u}_1 \subseteq \mathfrak{u}_2$, or $\neg x \in \mathfrak{a}_1 \cap \mathscr{C}(\mathbf{A}) \subseteq \mathfrak{a}_2 \cap \mathscr{C}(\mathbf{A})$. This shows $u \vee \neg x \in \mathfrak{a}_2$, whence $a \in \mathfrak{a}_2$. This completes the proof. $\square$

Following our usual approach, we endow $\mathcal{F}_{\mathbf{A}}^{\bowtie}$ with additional structure in a manner that conserves α's being an isomorphism. The next definition provides the appropriate topological structure.

Definition 8.2.26. *Let* $\mathbf{A} \in \mathsf{srDL}$*. For clopen up-sets* $U \subseteq \mathcal{S}(\mathscr{B}(\mathbf{A}))$, $V \subseteq \mathcal{S}(\mathscr{R}(\mathbf{A}))$*, define*

$$W_{(U,V)} = [(U \times V) \cup +(U \times \mathcal{S}(N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]) \cup \mathcal{S}(\mathscr{B}(\mathbf{A})) \times N_{\mathbf{A}}[V]^{\mathsf{c}})] \cap \mathcal{F}_{\mathbf{A}}^{\bowtie},$$

where $N_{\mathbf{A}}[V] = \{N_{\mathbf{A}}[\mathfrak{x}] : \mathfrak{x} \in V\}$*, and for a subset* $P \subseteq \mathcal{S}(\mathscr{B}(\mathbf{A})) \times \mathcal{S}(\mathscr{R}(\mathbf{A}))$*,* $+P = \{+p : p \in P\}$*.*

Remark 8.2.27. Let $\Delta\colon \mathcal{S}(\mathscr{R}(\mathbf{A})) \to \mathcal{S}(\mathscr{R}(\mathbf{A}))$ be the dual of the lattice homomorphism $N_{\mathbf{A}}$, i.e. $\Delta(\mathfrak{x}) = N_{\mathbf{A}}^{-1}[\mathfrak{x}]$. Then Δ is a closure operator on $\mathcal{S}(\mathscr{R}(\mathbf{A}))$, and we let

$$\mathcal{S}(\mathscr{R}(\mathbf{A}))_{\Delta} := \Delta[\mathcal{S}(\mathscr{R}(\mathbf{A}))] = \{\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A})) : \Delta(\mathfrak{x}) = \mathfrak{x}\}$$

be the set of Δ-fixed points. Defining a map $\beta\colon \mathcal{S}(\mathscr{R}(\mathbf{A}))_{\Delta} \to \mathcal{S}(N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})])$ by $\beta(\mathfrak{x}) = \mathfrak{x} \cap N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]$, one may obtain by an argument identical to that given in [4, Theorem 12 and Lemma 25] that β is an isomorphism of Priestley spaces when $\mathcal{S}(\mathscr{R}(\mathbf{A}))_{\Delta}$ is viewed as a subspace of $\mathcal{S}(\mathscr{R}(\mathbf{A}))$. The inverse morphism of β is given by $\mathfrak{x} \mapsto \Delta(\mathfrak{x})$.

Also, if $V \subseteq \mathcal{S}(\mathscr{R}(\mathbf{A}))$ is a clopen up-set, one may show that image

$$N_{\mathbf{A}}[V] = \{N_{\mathbf{A}}[\mathfrak{x}] : \mathfrak{x} \in V\}$$

under β^{-1} is

$$\Delta[V] = \{\mathfrak{x} \in V : \Delta(\mathfrak{x}) = \mathfrak{x}\} = V \cap \mathcal{S}(\mathscr{R}(\mathbf{A}))_{\Delta}.$$

From these observations, we may identify $\mathcal{S}(N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})])$ and $\mathcal{S}(\mathscr{R}(\mathbf{A}))_{\Delta}$, as well as $\Delta[V]$ and $N_{\mathbf{A}}[V]$, in the definition of the topology on $\mathcal{F}^{\bowtie}_{\mathbf{A}}$ offered above. Hence the sets $W_{(U,V)}$ may be rewritten in a manner that depends only on Δ, and not on $N_{\mathbf{A}}$.

Lemma 8.2.28. *Let* $\mathbf{A} \in \mathsf{srDL}$. *Then* $N_{\mathbf{A}}[\varphi_{\mathscr{R}(\mathbf{A})}(x)] = \varphi_{N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]}(N_{\mathbf{A}}(x))$ *for all* $x \in \mathscr{R}(\mathbf{A})$.

Proof. Let $\mathfrak{y} \in N_{\mathbf{A}}[\varphi_{\mathscr{R}(\mathbf{A})}(x)]$. Then there exists $\mathfrak{x} \in \varphi_{\mathscr{R}(\mathbf{A})}(x)$ such that $N_{\mathbf{A}}[\mathfrak{x}] = \mathfrak{y}$. Since $N_{\mathbf{A}}$ is a wdl-admissible map, we may show that $\mathfrak{y} = N_{\mathbf{A}}[\mathfrak{x}] \in \mathcal{S}(N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})])$. Also, $N_{\mathbf{A}}(x) \in N_{\mathbf{A}}[\mathfrak{x}] = \mathfrak{y}$ since $x \in \mathfrak{x}$. From this it follows that $\mathfrak{y} \in \varphi_{N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]}(N_{\mathbf{A}}(x))$, whence $N_{\mathbf{A}}[\varphi_{\mathscr{R}(\mathbf{A})}(x)] \subseteq \varphi_{N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]}(N_{\mathbf{A}}(x))$.

To prove the reverse inclusion, let $\mathfrak{y} \in \varphi_{N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]}(N_{\mathbf{A}}(x))$, and set $\mathfrak{x} = N_{\mathbf{A}}^{-1}[\mathfrak{y}]$. From $N_{\mathbf{A}}$ being a lattice homomorphism, we obtain $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$. Also, $N_{\mathbf{A}}(x) \in \mathfrak{y}$ implies $x \in N_{\mathbf{A}}^{-1}[\mathfrak{y}] = \mathfrak{x}$, whence $\mathfrak{x} \in \varphi_{\mathscr{R}(\mathbf{A})}(x)$. An easy argument shows $N_{\mathbf{A}}[\mathfrak{x}] = \mathfrak{y}$, from which the result follows. $\square$

Henceforth we consider $\mathcal{F}^{\bowtie}_{\mathbf{A}}$ endowed with the topology generated by the sets $W_{(U,V)}$ and $W^{\mathsf{c}}_{(U,V)}$, where $(U, V) \in \mathcal{AS}(\mathscr{B}(\mathbf{A})) \times \mathcal{AS}(\mathscr{R}(\mathbf{A}))$.

Lemma 8.2.29. *Let* $\mathbf{A} \in \mathsf{srDL}$. *Then* $\alpha_{\mathbf{A}}$ *is continuous.*

Proof. We will show that inverse image under $\alpha_{\mathbf{A}}$ of the subbasis elements $W_{(U,V)}$ and $W^{\mathsf{c}}_{(U,V)}$ are open. Let $U \subseteq \mathcal{S}(\mathscr{B}(\mathbf{A}))$ and $V \subseteq \mathcal{S}(\mathscr{R}(\mathbf{A}))$ be clopen up-sets. According to extended Priestley duality, the functions $\varphi_{\mathscr{B}(\mathbf{A})} \colon \mathscr{B}(\mathbf{A}) \to \mathcal{AS}(\mathscr{B}(\mathbf{A}))$ and $\varphi_{\mathscr{R}(\mathbf{A})} \colon \mathscr{R}(\mathbf{A}) \to \mathcal{AS}(\mathscr{R}(\mathbf{A}))$ are isomorphisms. Thus there are $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$ with $U = \varphi_{\mathscr{B}(\mathbf{A})}(u)$ and $V = \varphi_{\mathscr{R}(\mathbf{A})}(x)$. Set $a := (u \vee \neg x) \wedge (\neg u \vee x)$. We will prove $\alpha_{\mathbf{A}}^{-1}[W_{(U,V)}] = \varphi_{\mathbf{A}}(a)$.

For the forward inclusion, let $\mathfrak{a} \in \alpha_{\mathbf{A}}^{-1}[W_{(U,V)}]$ so that $\alpha_{\mathbf{A}}(\mathfrak{a}) \in W_{(U,V)}$. There are two cases.

Case 1: In this situation, $\mathfrak{a} \subseteq \mathfrak{a}^*$. $\alpha_{\mathbf{A}}(\mathfrak{a}) = (\mathfrak{a} \cap \mathscr{B}(\mathbf{A}), \mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \in U \times V$ and $\mathfrak{a} \cap \mathscr{B}(\mathbf{A}) \in \varphi_{\mathscr{B}(\mathbf{A})}(u)$, $\mathfrak{a} \cap \mathscr{R}(\mathbf{A}) \in \varphi_{\mathscr{R}(\mathbf{A})}(x)$. It follows that $u \in \mathfrak{a} \cap \mathscr{B}(\mathbf{A})$ and $x \in \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$, and $u, x \in \mathfrak{a}$ in particular. Since $\mathfrak{a}$ is an up-set, this yields $a = (u \vee \neg x) \wedge (\neg u \vee x) \in \mathfrak{a}$, so $\mathfrak{a} \in \varphi_{\mathbf{A}}(a)$.

Case 2: $\mathfrak{a}^* \subset \mathfrak{a}$. In this case, we have

$$\alpha_{\mathbf{A}}(\mathfrak{a}) = +(\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}), N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]),$$

where

$$\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}) \in U = \varphi_{\mathscr{B}(\mathbf{A})}(u), \text{ or}$$

$$N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})] \in N_{\mathbf{A}}[V]^{\mathsf{c}} = N_{\mathbf{A}}[\varphi_{\mathscr{R}(\mathbf{A})}(x)]^{\mathsf{c}}.$$

In the first situation, $\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}) \in \varphi_{\mathscr{B}(\mathbf{A})}(u)$ and $u \in \mathfrak{a}^*$. It follows that $u \in \mathfrak{a}$ (i.e., as $\mathfrak{a}$ and $\mathfrak{a}^*$ have the same ultrafilter from Lemma 8.2.19). Then $u \vee \neg x \in \mathfrak{a}$ as $\mathfrak{a}$ is an up-set. In the second situation, $N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})] \in N_{\mathbf{A}}[\varphi_{\mathscr{R}(\mathbf{A})}(x)]^{\mathsf{c}}$ and we have that $N_{\mathbf{A}}[\varphi_{\mathscr{R}(\mathbf{A})}(x)]^{\mathsf{c}} = \varphi_{N_{\mathbf{A}}[\mathscr{R}(\mathbf{A})]}(N_{\mathbf{A}}(x))^{\mathsf{c}}$ by Lemma 8.2.28. Thus $N_{\mathbf{A}}(x) \notin N_{\mathbf{A}}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]$. This implies $x \notin \mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})$, and as $x \in \mathscr{R}(\mathbf{A})$ we get $x \notin \mathfrak{a}^*$. Hence $\neg x \in \mathfrak{a}$ by the definition of $*$, and therefore $u \vee \neg x \in \mathfrak{a}$. As $\mathfrak{a}^* \subset \mathfrak{a}$, applying Lemma 8.2.21 yields $\mathfrak{a}^* \subseteq \mathscr{R}_{\mathfrak{u}_{\mathfrak{a}}} \subseteq \mathfrak{a}$. Thus $\mathscr{R}(\mathbf{A}) \subseteq \mathfrak{a}$ and $x \in \mathfrak{a}$, whence $\neg u \vee x \in \mathfrak{a}$ as $\mathfrak{a}$ is an up-set. This implies that both of $u \vee \neg x, \neg u \vee x \in \mathfrak{a}$, so $a = (u \vee \neg x) \wedge (\neg u \vee x) \in \mathfrak{a}$. We obtain $\mathfrak{a} \in \varphi_{\mathbf{A}}(a)$, and hence that $\alpha_{\mathbf{A}}^{-1}[W_{(U,V)}] \subseteq \varphi_{\mathbf{A}}(a)$.

For the backward inclusion, let $\mathfrak{a} \in \varphi_{\mathbf{A}}(a)$. Then $a = (u \vee \neg x) \wedge (\neg u \vee x) \in \mathfrak{a}$, whence $u \vee \neg x, \neg u \vee x \in \mathfrak{a}$. The primality of $\mathfrak{a}$ implies that the following two propositions hold: (1) Either $u \in \mathfrak{a}$ or $\neg x \in \mathfrak{a}$, and (2) either $\neg u \in \mathfrak{a}$ or $x \in \mathfrak{a}$. Again, there are two cases.

Case 1: $\mathfrak{a} \subseteq \mathfrak{a}^*$. Here Lemma 8.2.21 implies $\mathfrak{a} \subseteq \mathscr{R}_{\mathfrak{u}_\mathfrak{a}} \subseteq \mathfrak{a}^*$. Observe that since since $\neg x \in \mathscr{C}(\mathbf{A})$ and $\mathfrak{a} \subseteq \mathscr{R}_{\mathfrak{u}_\mathfrak{a}}$ we obtain $\neg x \notin \mathfrak{a}$, so by (1) we get $u \in \mathfrak{a}$. Then $\neg u \notin \mathfrak{a}$, whence by (2) we have $x \in \mathfrak{a}$. This implies $u, x \in \mathfrak{a}$, and therefore $\mathfrak{a} \cap \mathscr{B}(\mathbf{A}) \in \varphi_{\mathscr{B}(\mathbf{A})}(u)$ and $\mathfrak{a} \cap \mathscr{R}(\mathbf{A}) \in \varphi_{\mathscr{R}(\mathbf{A})}(x)$, so $\alpha_\mathbf{A}(\mathfrak{a}) \in U \times V$.

Case 2: $\mathfrak{a}^* \subset \mathfrak{a}$. We have

$$\alpha_\mathbf{A}(\mathfrak{a}) = +(\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}), N_\mathbf{A}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]).$$

By (1) either $u \in \mathfrak{a}$ or $\neg x \in \mathfrak{a}$. In the situation that $u \in \mathfrak{a}$, we get

$$\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}) = \mathfrak{a} \cap \mathscr{B}(\mathbf{A}) \in \varphi_{\mathscr{B}(\mathbf{A})}(u) = U,$$

whence $(\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}), N_\mathbf{A}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]) \in U \times \mathcal{S}(N_\mathbf{A}[\mathscr{R}(\mathbf{A})])$. On the other hand, if $\neg x \in \mathfrak{a}$, then $\neg\neg\neg x = \neg x$ implies $\neg\neg\neg x \in \mathfrak{a}$. This gives $N_\mathbf{A}(x) = \neg\neg x \notin \mathfrak{a}^*$. Hence $N_\mathbf{A}(x) \notin \mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})$, and thus $N_\mathbf{A}(x) \notin N_\mathbf{A}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]$, i.e.,

$$N_\mathbf{A}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})] \in \varphi_{N_\mathbf{A}[\mathscr{R}(\mathbf{A})]}(N_\mathbf{A}(x))^\mathsf{c} = N_\mathbf{A}[V]^\mathsf{c}.$$

It follows that

$$(\mathfrak{a}^* \cap \mathscr{B}(\mathbf{A}), N_\mathbf{A}[\mathfrak{a}^* \cap \mathscr{R}(\mathbf{A})]) \in \mathcal{S}(\mathscr{B}(\mathbf{A})) \times N_\mathbf{A}[V]^\mathsf{c},$$

so $\alpha_\mathbf{A}(\mathfrak{a}) \in +(U \times \mathcal{S}(N_\mathbf{A}[\mathscr{R}(\mathbf{A})]) \cup \mathcal{S}(\mathscr{B}(\mathbf{A})) \times N_\mathbf{A}[V]^\mathsf{c})$. This demonstrates that $\varphi_\mathbf{A}(a) = \alpha_\mathbf{A}^{-1}[W_{(U,V)}]$.

To finish the proof, note that since $\alpha_\mathbf{A}$ is a bijection we have

$$\alpha_\mathbf{A}^{-1}[W_{(U,V)}^\mathsf{c}] = (\alpha_\mathbf{A}^{-1}[W_{(U,V)}])^\mathsf{c} = \varphi_\mathbf{A}(a)^\mathsf{c}$$

when a is as above. Thus the $\alpha_{\mathbf{A}}$-inverse image of subbasis elements are open, and thus $\alpha_{\mathbf{A}}$ is continuous. $\square$

Remark 8.2.30. The proof given above shows more. Clopen subbasis elements of $\mathcal{S}(\mathbf{A})$ and $\mathcal{F}_{\mathbf{A}}^{\bowtie}$ precisely correspond under $\alpha_{\mathbf{A}}$, so since $\alpha_{\mathbf{A}}$ is an order isomorphism we have that all structure is transported from $\mathcal{S}(\mathbf{A})$ to $\mathcal{F}_{\mathbf{A}}^{\bowtie}$. Thus $\mathcal{F}_{\mathbf{A}}^{\bowtie}$ is a Priestley space that is isomorphic in Pries to $\mathcal{S}(\mathbf{A})$.

Example 8.2.31. *Let $A = \{-3,-2,-1,1,2,3\}$. If we view $\{1,2,3\}$ as the three-element Gödel algebra with order given by $1 < 2 < 3$ and residual $\to$, then we may make A into an srDL-algebra by defining the order by $-3 < -2 < -1 < 1 < 2 < 3$ and*

$$a \cdot b = \begin{cases} a \wedge b & a, b > 0 \\ -(a \to -b) & a > 0, b < 0 \\ -(b \to -a) & a < 0, b > 0 \\ -3 & a, b < 0 \end{cases}$$

Denote the resulting srDL-algebra by $\mathbf{A}$. Then

$$\mathscr{B}(\mathbf{A}^2) = \{(-3,-3), (-3,3), (3,-3), (3,3)\}$$

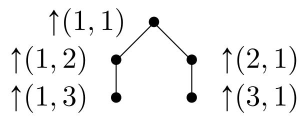
and

$$\mathscr{R}(\mathbf{A}^2) = {\uparrow}\{1,1\}.$$

It follows that $\mathcal{S}(\mathscr{B}(\mathbf{A}^2)) = \{\mathfrak{u}, \mathfrak{v}\}$ is the two-element Stone space, where

$$\mathfrak{u} = {\uparrow}(-3,3) \cap \mathscr{B}(\mathbf{A}^2) \text{ and } \mathfrak{v} = {\uparrow}(3,-3) \cap \mathscr{B}(\mathbf{A}^2).$$

The Priestley space of $\mathscr{R}(\mathbf{A}^2)$ has labeled Hasse diagram



We may obtain $\mathcal{F}^{\bowtie}_{\mathbf{A^2}}$ *by determining which proper* $\mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}^2))$ *are fixed by* $\mathfrak{u}$ *(respectively* $\mathfrak{v}$*). For this, note that*

$$\mu_{(-3,3)}(\uparrow(1,3)) = \uparrow(1,1) = \mathscr{R}(\mathbf{A}^2)$$

$$\mu_{(-3,3)}(\uparrow(1,2)) = \uparrow(1,1) = \mathscr{R}(\mathbf{A}^2)$$

$$\mu_{(-3,3)}(\uparrow(2,1)) = \uparrow(2,1)$$

$$\mu_{(-3,3)}(\uparrow(3,1)) = \uparrow(1,1)$$

and

$$\mu_{(3,-3)}(\uparrow(1,3)) = \uparrow(1,3)$$

$$\mu_{(3,-3)}(\uparrow(1,2)) = \uparrow(1,2)$$

$$\mu_{(3,-3)}(\uparrow(2,1)) = \uparrow(1,1) = \mathscr{R}(\mathbf{A}^2)$$

$$\mu_{(3,-3)}(\uparrow(3,1)) = \uparrow(1,1) = \mathscr{R}(\mathbf{A}^2)$$

It follows that $\mathfrak{u}$ *fixes* $\uparrow(1,3)$ *and* $\uparrow(1,2)$*, whereas* $\mathfrak{v}$ *fixes* $\uparrow(2,1)$ *and* $\uparrow(3,1)$*. To get* $\mathcal{F}^{\bowtie}_{\mathbf{A^2}}$*, we append a copy of each* $\mathfrak{x}$ *below* $\mathfrak{u}$ *(respectively* $\mathfrak{v}$*) if it is fixed by* $\mathfrak{u}$ *(respectively* $\mathfrak{v}$*), and a fresh copy of the poset obtained in this way is then reflected upward, as pictured in Figure 8.1.*

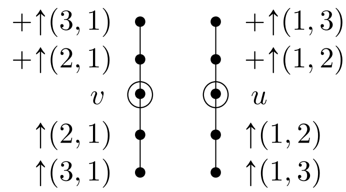


Figure 8.1: Labeled Hasse diagram for $\mathcal{F}^{\bowtie}_{\mathbf{A^2}}$.

8.3 Filter multiplication in srDL

Section 8.2 lays out the necessary ingredients to construct the Priestley space of (the lattice reduct of) an srDL-algebra $\mathbf{A}$ from the Priestley duals of $\mathcal{S}(\mathscr{B}(\mathbf{A}))$ and $\mathcal{S}(\mathscr{R}(\mathbf{A}))$. This section attends to characterizing the filter multiplication on an srDL-algebra in terms of these components, in particular defining a ternary relation on $\mathcal{F}^{\bowtie}_{\mathbf{A}}$ that makes $\alpha_{\mathbf{A}}$ into an isomorphism of MTL^{τ}. Recall that the site of $\mathfrak{u}$ in $\mathbf{A}$ is the set

$$\mathcal{S}_{\mathfrak{u}} = \{\mathfrak{a} \in \mathcal{S}(A) : \mathfrak{u} = \mathfrak{u}_{\mathfrak{a}}\}.$$

Our first lemma permits us to focus on the sets $\mathcal{S}_{\mathfrak{u}}$ in our analysis of filter multiplication.

Lemma 8.3.1. *Let $\mathbf{A} \in \mathsf{srDL}$, and let $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}(A)$. Then $\mathfrak{u}_{\mathfrak{a}} \neq \mathfrak{u}_{\mathfrak{b}}$ implies that $\mathfrak{a} \bullet \mathfrak{b} = A$.*

Proof. Suppose without loss of generality that there exists $u \in \mathfrak{u}_{\mathfrak{a}} \subseteq \mathfrak{a}$ with $u \notin \mathfrak{u}_{\mathfrak{b}}$. Then $\neg u \in \mathfrak{u}_{\mathfrak{b}} \subseteq \mathfrak{b}$ since $\mathfrak{u}_{\mathfrak{b}}$ is an ultrafilter. Therefore $u \cdot \neg u = u \wedge \neg u = 0 \in \mathfrak{a} \bullet \mathfrak{b}$, whence $\mathfrak{a} \bullet \mathfrak{b} = A$. □

Lemma 8.3.2. *Let $\mathbf{A} \in \mathsf{srDL}$ and let $\mathfrak{a} \in \mathcal{S}(A)$ with $\mathscr{R}(\mathbf{A}) \subseteq \mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{a}^{**}$.*

Proof. The definition of * provides that $a \in \mathfrak{a}^{**}$ if and only if $\neg\neg a \in \mathfrak{a}$, whence it is necessary and sufficient to show $a \in \mathfrak{a}$ if and only if $\neg\neg a \in \mathfrak{a}$. Since $\mathfrak{a}$ is an up-set, the identity $a \leqslant \neg\neg a$ implies that $\neg\neg a \in \mathfrak{a}$ for each $a \in \mathfrak{a}$. Conversely, suppose that $\neg\neg a \in \mathfrak{a}$. Using Equation 2.3.1, write $a = (u \wedge \neg x) \vee (\neg u \wedge x)$ for some $u \in \mathscr{B}(\mathbf{A})$ and $x \in \mathscr{R}(\mathbf{A})$. Then

$$\begin{aligned}\neg\neg a &= (u \wedge \neg\neg\neg x) \vee (\neg u \wedge \neg\neg x)\\ &= (u \wedge \neg x) \vee (\neg u \wedge \neg\neg x) \in \mathfrak{a}\end{aligned}$$

Primality yields that $u \wedge \neg x \in \mathfrak{a}$ or $\neg u \wedge \neg\neg x \in \mathfrak{a}$. In the first case, $a \in \mathfrak{a}$ follows because $u \wedge \neg x \leqslant a$. In the second case, $\neg u \wedge \neg\neg x \leqslant \neg u$ implies that $\neg u \in \mathfrak{a}$, so as $x \in \mathscr{R}(\mathbf{A}) \subseteq \mathfrak{a}$ we get $\neg u \wedge x \in \mathfrak{a}$. Thus by $\neg u \wedge x \leqslant a$ and $\mathfrak{a}$ being being an up-set, we have $a \in \mathfrak{a}$. This proves the claim. □

The following provides a characterization of filter multiplication on any srDL-algebra.

Lemma 8.3.3. *Let $\mathbf{A} \in \mathsf{srDL}$, let $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$, and let $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}_{\mathfrak{u}}$. Denote by $\bullet^{\mathscr{R}(\mathbf{A})}$ and $\Rightarrow^{\mathscr{R}(\mathbf{A})}$ the operations on $\mathcal{S}(\mathscr{R}(\mathbf{A}))$ defined as in Section 4.1. Then the following hold.*

1. *If $\mathfrak{a}, \mathfrak{b} \subseteq \mathscr{R}_{\mathfrak{u}}$, then $\mathfrak{a} \bullet \mathfrak{b} = \langle \mathfrak{u} \cup [(\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \bullet^{\mathscr{R}(\mathbf{A})} (\mathfrak{b} \cap \mathscr{R}(\mathbf{A}))] \rangle$.*
2. *If $\mathfrak{a} \subseteq \mathfrak{b}^* \subseteq \mathscr{R}_{\mathfrak{u}} \subseteq \mathfrak{b}$, then $\mathfrak{a} \bullet \mathfrak{b} = \langle \mathfrak{u} \cup [((\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \Rightarrow^{\mathscr{R}(\mathbf{A})} (\mathfrak{b}^* \cap \mathscr{R}(\mathbf{A})))] \rangle^*$.*
3. *If none of $\mathfrak{a}, \mathfrak{b} \subseteq \mathscr{R}_{\mathfrak{u}}$, $\mathfrak{a} \subseteq \mathfrak{b}^* \subseteq \mathscr{R}_{\mathfrak{u}} \subseteq \mathfrak{b}$, or $\mathfrak{b} \subseteq \mathfrak{a}^* \subseteq \mathscr{R}_{\mathfrak{u}} \subseteq \mathfrak{a}$ hold, then $\mathfrak{a} \bullet \mathfrak{b} = A$.*

Proof. To prove (1), note that $\mathfrak{a} \bullet \mathfrak{b} \in \mathcal{S}_{\mathfrak{u}}$, and that $\mathfrak{a} \bullet \mathfrak{b} \subseteq \mathscr{R}_{\mathfrak{u}}$ since $\bullet$ is order-preserving and $\mathscr{R}_{\mathfrak{u}} \bullet \mathscr{R}_{\mathfrak{u}} = \mathscr{R}_{\mathfrak{u}}$. We will show that

$$\mathfrak{a} \bullet \mathfrak{b} \cap \mathscr{R}(\mathbf{A}) = (\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \bullet^{\mathscr{R}(\mathbf{A})} (\mathfrak{b} \cap \mathscr{R}(\mathbf{A})).$$

For this, let $c \in \mathfrak{a} \bullet \mathfrak{b} \cap \mathscr{R}(\mathbf{A})$. Then $c \in \mathscr{R}(\mathbf{A})$, and there exist $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ with $a \cdot b \leqslant c$. From $a, c \leqslant a \vee c$, $b, c \leqslant b \vee c$, and $\mathfrak{a}, \mathfrak{b}, \mathscr{R}(\mathbf{A})$ being up-sets, we infer $a \vee c \in \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$ and $b \vee c \in \mathfrak{b} \cap \mathscr{R}(\mathbf{A})$. Note that

$$(a \vee c) \cdot (b \vee c) = ab \vee ac \vee bc \vee c^2 \leqslant c,$$

so $c \in (\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \bullet^{\mathscr{R}(\mathbf{A})} (\mathfrak{b} \cap \mathscr{R}(\mathbf{A}))$. Hence $\mathfrak{a} \bullet \mathfrak{b} \cap \mathscr{R}(\mathbf{A}) \subseteq (\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \bullet^{\mathscr{R}(\mathbf{A})} (\mathfrak{b} \cap \mathscr{R}(\mathbf{A}))$.

To obtain the other inclusion, let $c \in (\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \bullet^{\mathscr{R}(\mathbf{A})} (\mathfrak{b} \cap \mathscr{R}(\mathbf{A}))$. Then there exist $a \in \mathfrak{a} \cap \mathscr{R}(\mathbf{A})$, $b \in \mathfrak{b} \cap \mathscr{R}(\mathbf{A})$, with $a \cdot b \leqslant c$. Note that $a \cdot b \in \mathscr{R}(\mathbf{A})$ as $\mathscr{R}(\mathbf{A})$ is closed under $\cdot$, so $c \in \mathscr{R}(\mathbf{A})$. Thus $a \in \mathfrak{a}$, $b \in \mathfrak{b}$, and $c \in \mathscr{R}(\mathbf{A})$ give $c \in \mathfrak{a} \bullet \mathfrak{b} \cap \mathscr{R}(\mathbf{A})$, whence we obtain equality.

To prove (2), let $\mathfrak{a} = \langle \mathfrak{u} \cup \mathfrak{y} \rangle$ and $\mathfrak{b}^* = \langle \mathfrak{u} \cup \mathfrak{x} \rangle$, so we have $\mathfrak{b} = \mathfrak{b}^{**} = \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$ by Lemma 8.3.2. From $\mathfrak{a} \subseteq \mathfrak{b}^*$ we have $\mathfrak{a} \cap \mathscr{R}(\mathbf{A}) \subseteq \mathfrak{b}^* \cap \mathscr{R}(\mathbf{A})$. It follows that $\{1\} \bullet^{\mathscr{R}(\mathbf{A})} (\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \subseteq \mathfrak{b}^* \cap \mathscr{R}(\mathbf{A})$, and applying Lemma 4.1.1(2) we get a prime filter $\mathfrak{z} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$ with $\mathfrak{z} \bullet^{\mathscr{R}(\mathbf{A})} (\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \subseteq \mathfrak{b}^* \cap \mathscr{R}(\mathbf{A})$. Consequently,

$$\mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x} = (\mathfrak{a} \cap \mathscr{R}(\mathbf{A})) \Rightarrow^{\mathscr{R}(\mathbf{A})} (\mathfrak{b}^* \cap \mathscr{R}(\mathbf{A})) \neq \varnothing,$$

and thus $\mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x} \in \mathcal{S}(\mathscr{R}(\mathbf{A}))$, i.e., $\Rightarrow^{\mathscr{R}(\mathbf{A})}$ is defined in this situation. We claim that

$$\langle \mathfrak{u} \cup \mathfrak{y} \rangle \bullet \langle \mathfrak{u} \cup \mathfrak{x} \rangle^* = \langle \mathfrak{u} \cup (\mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x}) \rangle^*.$$

Let $a \in \langle \mathfrak{u} \cup \mathfrak{y} \rangle \bullet \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$. Then there exist $w \in \langle \mathfrak{u} \cup \mathfrak{y} \rangle$ and $z \in \langle \mathfrak{u} \cup \mathfrak{x} \rangle^*$ such that $z \cdot w \leqslant a$. By Lemma 8.2.16 this implies that there exist $b, b' \in \mathfrak{u}$, $\neg\neg c \in \mathscr{R}(\mathbf{A}) \backslash \mathfrak{x}$, and $d \in \mathfrak{y}$ with $b \wedge \neg c \leqslant z$ and $b' \wedge d \leqslant w$. Hence $(b \wedge \neg c) \cdot (b' \wedge d) \leqslant z \cdot w \leqslant a$. By checking on directly indecomposable components, we get $(b \wedge \neg c) \cdot (b' \wedge d) = (b \cdot b') \wedge (\neg c \cdot d)$. Now observe that $\neg c \cdot d \in \mathscr{C}(\mathbf{A})$, whence there is $z \in \mathscr{R}(\mathbf{A})$ with $\neg z = \neg c \cdot d$. We will show that $z \notin N_{\mathbf{A}}[\mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x}]$, from which it will follow that $a \in \langle \mathfrak{u} \cup (\mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x}) \rangle^*$

by Lemma 8.2.16. Toward a contradiction, assume $z \in \mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x}$. Then $z \cdot y \in \mathfrak{x}$ for every $y \in \mathfrak{y}$, and in particular $z \cdot d \in \mathfrak{x}$. We have $z \cdot \neg z = z \cdot (\neg c \cdot d) = 0$, and therefore $z \cdot d \leqslant \neg\neg c$ and $\neg\neg c \in \mathfrak{x}$. This contradicts $\neg\neg c \notin \mathfrak{x}$. This yields the left-to-right inclusion.

For the other inclusion, let $a \in \langle \mathfrak{u} \cup (\mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x})\rangle^*$. Then by Lemma 8.2.16 there exists $u \in \mathfrak{u}$, $\neg\neg z \notin N_{\mathbf{A}}[\mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x}]$ such that $u \wedge \neg z \leqslant a$. As $\neg\neg z \notin N_{\mathbf{A}}[\mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x}]$, it follows that $z \notin \mathfrak{y} \Rightarrow^{\mathscr{R}(\mathbf{A})} \mathfrak{x}$, whence there exists $y \in \mathfrak{y}$ such that $yz \notin \mathfrak{x}$. This implies that $yz \notin \langle \mathfrak{u} \cup \mathfrak{x}\rangle$, and since $\mathfrak{b}^* = \langle \mathfrak{u} \cup \mathfrak{x}\rangle$ we have that $\neg\neg(yz) \notin \langle \mathfrak{u} \cup \mathfrak{x}\rangle$. To see this, note that by the definition of $*$,

$$\neg\neg x \in \mathfrak{b}^* \iff \neg\neg\neg x \notin \mathfrak{b} \iff \neg x \notin \mathfrak{b} \iff x \in \mathfrak{b}^*.$$

Now since $\neg\neg(yz) \notin \langle \mathfrak{u} \cup \mathfrak{x}\rangle$, it follows that $\neg(yz) \in \langle \mathfrak{u} \cup \mathfrak{x}\rangle^*$. Observe that

$$\begin{aligned} \neg(yz) &= (yz) \to 0 \\ &= y \to (z \to 0) \\ &= y \to \neg z \end{aligned}$$

Thus $y \to \neg z \in \langle \mathfrak{u} \cup \mathfrak{x}\rangle^*$, and thus $\neg z \in \langle \mathfrak{u} \cup \mathfrak{y}\rangle \bullet \langle \mathfrak{u} \cup \mathfrak{x}\rangle^*$ as $y(y \to \neg z) \leqslant \neg z$. Since $u \in \langle \mathfrak{u} \cup \mathfrak{y}\rangle \bullet \langle \mathfrak{u} \cup \mathfrak{x}\rangle^*$, we obtain that $u \wedge \neg z \in \langle \mathfrak{u} \cup \mathfrak{y}\rangle \bullet \langle \mathfrak{u} \cup \mathfrak{x}\rangle^*$, from which we get that a is contained in the latter set as $u \wedge \neg z \leqslant a$. This gives the reverse inclusion, yielding equality and (2).

To prove (3), observe that $\mathfrak{a} \nsubseteq \mathfrak{b}^*$ and $\mathfrak{b} \nsubseteq \mathfrak{a}^*$ follow from the hypothesis. Lemma 4.1.8 asserts that $\mathfrak{c}^*$ is the largest element of $\mathcal{S}(A)$ such that $\mathfrak{c} \bullet \mathfrak{c}^* \neq A$, so we get $\mathfrak{a} \bullet \mathfrak{b} = A$. $\square$

Lemma 8.3.3 offers us a complete description of the partial operation $\bullet$ on $\mathcal{S}(A)$ for $\mathbf{A} \in \mathsf{srDL}$, showing how to identify this operation in terms of the operation $\bullet^{\mathscr{R}(\mathbf{A})}$ and partial operation $\Rightarrow^{\mathscr{R}(\mathbf{A})}$ on $\mathcal{S}(\mathscr{R}(\mathbf{A}))$. The next corollary rephrases Lemma 8.3.3 in the context of $\mathcal{F}^{\bowtie}_{\mathbf{A}}$, using Proposition 8.2.14 and the isomorphism $\alpha_{\mathbf{A}}$ to transport structure.

Corollary 8.3.4. *Let $\mathbf{A} \in \mathsf{srDL}$ and let $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}_{\mathfrak{u}}$ for some $\mathfrak{u} \in \mathcal{S}(\mathscr{B}(\mathbf{A}))$. Then the following hold.*

1. *If $\alpha_{\mathbf{A}}(\mathfrak{a}) = (\mathfrak{u}, \mathfrak{x})$, and $\alpha_{\mathbf{A}}(\mathfrak{b}) = (\mathfrak{u}, \mathfrak{y})$ are in $\mathcal{F}_{\mathbf{A}}$, then $\alpha_{\mathbf{A}}(\mathfrak{a} \bullet \mathfrak{b}) = (\mathfrak{u}, \mathfrak{x} \bullet^{\mathscr{R}(\mathbf{A})} \mathfrak{y})$.*
2. *If $\alpha_{\mathbf{A}}(\mathfrak{a}) = (\mathfrak{u}, \mathfrak{x}) \in \mathcal{F}_{\mathbf{A}}$ and $\alpha_{\mathbf{A}}(\mathfrak{b}) = +(\mathfrak{u}, \mathfrak{y}) \in \mathcal{F}^{\partial}_{\mathbf{A}}$ with $(\mathfrak{u}, \mathfrak{x}) \sqsubseteq (\mathfrak{u}, N^{-1}_{\mathbf{A}}[\mathfrak{y}])$, then $\alpha_{\mathbf{A}}(\mathfrak{a} \bullet \mathfrak{b}) = +(\mathfrak{u}, \mathfrak{x} \Rightarrow^{\mathscr{R}(\mathbf{A})} N^{-1}_{\mathbf{A}}[\mathfrak{y}])$.*

In light of the facts assembled above, for $\mathbf{A} \in \mathsf{srDL}$ we may define a partial operation $\circ$ on $\mathcal{F}^{\bowtie}_{\mathbf{A}}$ by

1. $(\mathfrak{u}, \mathfrak{x}) \circ (\mathfrak{u}, \mathfrak{y}) = (\mathfrak{u}, \mathfrak{x} \bullet^{\mathscr{R}(\mathbf{A})} \mathfrak{y})$ for any $(\mathfrak{u}, \mathfrak{x}), (\mathfrak{u}, \mathfrak{y}) \in \mathcal{F}_{\mathbf{A}}$.
2. $(\mathfrak{u}, \mathfrak{x}) \circ +(\mathfrak{u}, \mathfrak{y}) = +(\mathfrak{u}, \mathfrak{x} \Rightarrow^{\mathscr{R}(\mathbf{A})} N^{-1}_{\mathbf{A}}[\mathfrak{y}])$ for any $(\mathfrak{u}, \mathfrak{x}) \in \mathcal{F}_{\mathbf{A}}$, $+(\mathfrak{u}, \mathfrak{y}) \in \mathcal{F}^{\partial}_{\mathbf{A}}$ with $(\mathfrak{u}, \mathfrak{x}) \sqsubseteq (\mathfrak{u}, N^{-1}_{\mathbf{A}}[\mathfrak{y}])$.
3. $+(\mathfrak{u}, \mathfrak{y}) \circ (\mathfrak{u}, \mathfrak{x}) = +(\mathfrak{u}, \mathfrak{x} \Rightarrow^{\mathscr{R}(\mathbf{A})} N^{-1}_{\mathbf{A}}[\mathfrak{y}])$ for any $(\mathfrak{u}, \mathfrak{x}) \in \mathcal{F}_{\mathbf{A}}$, $+(\mathfrak{u}, \mathfrak{y}) \in \mathcal{F}^{\partial}_{\mathbf{A}}$ with $(\mathfrak{u}, \mathfrak{x}) \sqsubseteq (\mathfrak{u}, N^{-1}_{\mathbf{A}}[\mathfrak{y}])$.
4. $\circ$ undefined otherwise.

If $\mathbf{A} \in \mathsf{srDL}$ and $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}(A)$, then $\mathfrak{a} \bullet \mathfrak{b}$ is defined if and only if $\alpha_{\mathbf{A}}(\mathfrak{a}) \circ \alpha_{\mathbf{A}}(\mathfrak{b})$ is defined, when this occurs $\alpha_{\mathbf{A}}(\mathfrak{a} \bullet \mathfrak{b}) = \alpha_{\mathbf{A}}(\mathfrak{a}) \circ \alpha_{\mathbf{A}}(\mathfrak{b})$. By augmenting $\mathcal{F}^{\bowtie}_{\mathbf{A}}$ with (the ternary relation associated to) $\circ$, the map $\alpha_{\mathbf{A}}$ becomes as isomorphism of MTL^{τ} and not just an isomorphism in Pries.

8.4 Dual quadruples and the dual construction

We now offer our dualized account of the construction of [1]. The following definitions rephrase key notions from this chapter in more abstract terms.

Definition 8.4.1. *We call a structure* $(\mathbf{S}, \mathbf{X}, \Upsilon, \Delta)$ *a* dual quadruple *if it satisfies the following.*

1. $\mathbf{S}$ *is a Stone space.*

2. $\mathbf{X}$ *is an object of* GMTL^{τ}*.*

3. $\Upsilon = \{\upsilon_U\}_{U \in \mathcal{A}(\mathbf{S})}$ *is an indexed family of* GMTL^{τ}*-morphisms* $\upsilon_U : \mathbf{X} \to \mathbf{X}$ *such that the map* $\vee_e \colon \mathcal{A}(\mathbf{S}) \times \mathcal{A}(\mathbf{X}) \to \mathcal{A}(\mathbf{X})$ *defined by*

$$\vee_e(U, V) = \upsilon_U^{-1}[V]$$

   is an external join.

4. $\Delta : \mathbf{X} \to \mathbf{X}$ *is a continuous closure operator such that* $R(x, y, z)$ *implies* $R(\Delta x, \Delta y, \Delta z)$*.*

Definition 8.4.2. *Let* $(\mathbf{S}, \mathbf{X}, \Upsilon, \Delta)$ *be a dual quadruple. We say that* $u \in S$ fixes $x \in X$ *if for every* $U \subseteq S$ *clopen with* $u \notin U$, $\upsilon_U(x) = x$*.*

Definition 8.4.3. *Let* $(\mathbf{S}, \mathbf{X}, \Upsilon, \Delta)$ *be a dual quadruple. Define*

$$\mathcal{D} = \{(u, x) \in S \times X : u \text{ fixes } x\}$$

$$\mathcal{D}^{\partial} = \{+(u, \Delta(x)) : (u, x) \in \mathcal{D}, x \neq \top\}$$

$$T = \mathcal{D} \cup \mathcal{D}^{\partial}.$$

Furthermore, define a partial order $\sqsubseteq$ *on* T *by* $p \sqsubseteq q$ *if and only if*

1. $p = (u, x)$ *and* $q = (v, y)$ *for some* $(u, x), (v, y) \in D$ *with* $u = v$ *and* $x \leqslant y$,
2. $p = +(u, x)$ *and* $q = +(v, y)$ *for some* $+(u, x), +(v, y) \in D^{\partial}$ *with* $u = v$ *and* $y \leqslant x$, *or*
3. $p = (u, x)$ *and* $q = +(v, y)$ *for some* $(u, x) \in D$, $(v, y) \in D^{\partial}$ *with* $u = v$.

For every $U \in \mathcal{A}(\mathbf{S})$, $V \in \mathcal{A}(\mathbf{X})$, *define*

$$W_{(U,V)} = [(U \times V) \cup +(U \times \Delta[X] \cup S \times \Delta[V]^{\mathsf{c}})] \cap T,$$

Let $\mathbf{S} \otimes_{\Upsilon}^{\Delta} \mathbf{X}$ *be the partially-ordered topological space with the order given above, and the topology generated by the subbase consisting of the sets* $W_{(U,V)}$ *and* $W_{(U,V)}^{\mathsf{c}}$. *Additionally, define a partial operation* $\circ$ *on* $\mathbf{S} \otimes_{\Upsilon}^{\Delta} \mathbf{X}$ *as follows, where* $\bullet$ *and* $\Rightarrow$ *denote the partial operations on* $\mathbf{X}$ *arising as in Section 4.1.*

1. $(u, x) \circ (u, y) = (u, x \bullet y)$ *for any* $(u, x), (u, y) \in \mathcal{F}_{\mathbf{A}}$.
2. $(u, x) \circ +(u, y) = +(u, x \Rightarrow \Delta(y))$ *for any* $(u, x) \in \mathcal{F}_{\mathbf{A}}$, $+(u, y) \in \mathcal{F}_{\mathbf{A}}^{\partial}$ *with* $(u, x) \sqsubseteq (u, \Delta(y))$.
3. $+(u, y) \circ (u, x) = +(u, x \Rightarrow \Delta(y))$ *for any* $(u, x) \in \mathcal{F}_{\mathbf{A}}$, $+(u, y) \in \mathcal{F}_{\mathbf{A}}^{\partial}$ *with* $(u, x) \sqsubseteq (u, \Delta(y))$.
4. $\circ$ *is undefined otherwise.*

We lastly expand $\mathbf{S} \otimes_{\Upsilon}^{\Delta} \mathbf{X}$ *by the ternary relation* R *defined by* $R(p, q, r)$ *if and only if* $p \circ q$ *exists and* $p \circ q \sqsubseteq r$.

Theorem 8.4.4. *Let* $(\mathbf{S}, \mathbf{X}, \Upsilon, \Delta)$ *be a dual quadruple. Then* $\mathbf{S} \otimes_{\Upsilon}^{\Delta} \mathbf{X}$ *is the extended Priestley dual of some srDL-algebra.*

Proof. Extended Priestley duality guarantees that there exist a Boolean algebra $\mathbf{B}$ and a GMTL-algebra $\mathbf{A}$ with $\mathbf{S} \cong \mathcal{S}(\mathbf{B})$ and $\mathbf{X} \cong \mathcal{S}(\mathbf{A})$. For the sake of simplicity of exposition, we identify these spaces. As $\mathcal{S}$ is full and Δ is a Priestley space morphism, there is a lattice homomorphism $N\colon \mathbf{A} \to \mathbf{A}$ such that $\mathcal{S}(N) = \Delta$. We will show that N is a wdl-admissible map on $\mathbf{A}$.

First, N is a closure operator: To get that N is expanding, suppose on the contrary $x \in A$ with $x \nleqslant N(x)$. Then there exists a prime filter $\mathfrak{x}$ of $\mathbf{A}$ such that $x \in \mathfrak{x}$ and $N(x) \notin \mathfrak{x}$ by the prime ideal theorem for distributive lattices. This implies that $x \in \mathfrak{x}$ and $x \notin N^{-1}[\mathfrak{x}] = \Delta(\mathfrak{x})$, contradicting Δ being expanding. It follows that $x \leqslant N(x)$ for all $x \in A$. N is idempotent by a proof similar to the one just given, and N is isotone because it is a lattice homomorphism. It follows that N is a closure operator.

Second, N is a nucleus: Let $x, y \in A$. We show that $N(x)N(y) \leqslant N(xy)$, and for this we assume on the contrary that there exists a prime filter $\mathfrak{z}$ of $\mathbf{A}$ with $N(x)N(y) \in \mathfrak{z}$ and $N(xy) \notin \mathfrak{z}$. This implies that $\uparrow N(x) \bullet \uparrow N(y) \subseteq \mathfrak{z}$. From Lemma 4.1.1(2) we obtain prime filters $\mathfrak{x}$ and $\mathfrak{y}$ with $N(x) \in \mathfrak{x}$, $N(y) \in \mathfrak{y}$ and $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. It follows that $R(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$, so $\Delta(\mathfrak{x}) \bullet \Delta(\mathfrak{y}) \subseteq \Delta(\mathfrak{z})$. But this is a contradiction to $N(xy) \notin \mathfrak{z}$ because $x \in N^{-1}[\mathfrak{x}] = \Delta(\mathfrak{x})$ and $y \in N^{-1}[\mathfrak{y}] = \Delta(\mathfrak{y})$, whence $xy \in \Delta(\mathfrak{z})$. Hence N is a wdl-admissible map.

For the rest, observe that by extended Stone-Priestley duality we have that for each $u \in \mathbf{B}$, there exists a homomorphism $\nu_u\colon \mathbf{A} \to \mathbf{A}$ such that $\mathcal{S}(\nu_u) = \upsilon_{\varphi_{\mathbf{B}}(u)}$. Define for each $u \in B$, $x \in A$, a map $\vee_e$ by

$$u \vee_e x = \nu_u(x).$$

We prove that $\vee_e$ is an external join. For this, observe that for all $x \in A$, $u \in B$, and $\mathfrak{x} \in \mathcal{S}(A)$,

$$\begin{aligned}
\mathfrak{x} \in \upsilon^{-1}_{\varphi_{\mathbf{B}}(u)}[\varphi_{\mathbf{A}}(x)] &\iff \upsilon_{\varphi_{\mathbf{B}}(u)}(\mathfrak{x}) \in \varphi_{\mathbf{A}}(x) \\
&\iff \nu_u^{-1}[\mathfrak{x}] \in \varphi_{\mathbf{A}}(x) \\
&\iff x \in \nu_u^{-1}[\mathfrak{x}] \\
&\iff \nu_u(x) \in \mathfrak{x} \\
&\iff \mathfrak{x} \in \varphi_{\mathbf{A}}(\nu_u(x)).
\end{aligned}$$

This provides $\upsilon^{-1}_{\varphi_{\mathbf{B}}(u)}[\varphi_{\mathbf{A}}(x)] = \varphi_{\mathbf{A}}(\nu_u(x))$. From this and Definition 8.4.1(3), we may readily show that $\vee_e$ satisfies condition (V1), (V2), and (V3) of Definition 8.1.1. For instance, for every $x \in A$, the map defined by $\lambda_x(u) = u \vee_e x$ gives a lattice homomorphism from $\mathbf{B} \to \mathbf{A}$ (as in (V1)). To see this, observe that

$$\begin{aligned}
\varphi_{\mathbf{A}}(\lambda_x(u \vee v)) &= \varphi_{\mathbf{A}}(\nu_{u \vee v}(x)) \\
&= \upsilon^{-1}_{\varphi_{\mathbf{B}}(u \vee v)}[\varphi_{\mathbf{A}}(x)] \\
&= \upsilon^{-1}_{\varphi_{\mathbf{B}}(u) \cup \varphi_{\mathbf{B}}(v)}[\varphi_{\mathbf{A}}(x)] \\
&= \upsilon^{-1}_{\varphi_{\mathbf{B}}(u)}[\varphi_{\mathbf{A}}(x)] \cup \upsilon^{-1}_{\varphi_{\mathbf{B}}(v)}[\varphi_{\mathbf{A}}(x)] \\
&= \varphi_{\mathbf{A}}(\nu_u(x)) \cup \varphi_{\mathbf{A}}(\nu_v(x)) \\
&= \varphi_{\mathbf{A}}(\lambda_x(u)) \cup \varphi_{\mathbf{A}}(\lambda_x(v)),
\end{aligned}$$

whence $\lambda_x(u \vee v) = \lambda_x(u) \vee \lambda_x(v)$ for every $x \in A$, $u, v \in B$. The other components of Definition 8.1.1(V1,V2,V3) may be checked by similar reasoning, using the assumption that $(U, V) \mapsto \upsilon_U^{-1}[V]$ is an external join. Hence $(\mathbf{B}, \mathbf{A}, \vee_e, N)$ is an algebraic quadruple. It follows that $\mathbf{S} \otimes^{\Delta}_{\curlyvee} \mathbf{X}$ is the extended Priestley space of $\mathbf{B} \otimes^N_e \mathbf{A}$ by construction. $\square$

Theorem 8.4.5. *Let* $\mathbf{Y}$ *be the extended Priestley dual of an srDL-algebra. Then there exists a dual quadruple* $(\mathbf{S}, \mathbf{X}, \Upsilon, \Delta)$ *with* $\mathbf{Y} \cong \mathbf{S} \otimes_{\Upsilon}^{\Delta} \mathbf{X}$*.*

Proof. Let $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 1, 0) \in \mathsf{srDL}$ with $\mathbf{Y} = \mathcal{S}(\mathbf{A})$, and let $\mathbf{S} := \mathcal{S}(\mathscr{B}(\mathbf{A}))$ and $\mathbf{X} := \mathcal{S}(\mathscr{R}(\mathbf{A}))$. Define $\Delta \colon \mathbf{X} \to \mathbf{X}$ by

$$\Delta(\mathfrak{x}) = \{x \in \mathscr{R}(\mathbf{A}) : \neg\neg x \in \mathfrak{x}\}.$$

Moreover, for each $U \in \mathcal{A}(\mathbf{S})$ define $\upsilon_U \colon \mathbf{X} \to \mathbf{X}$ by

$$\upsilon_U(\mathfrak{x}) = \mu_{\varphi^{-1}(U)}(\mathfrak{x}) = \{x \in \mathscr{R}(\mathbf{A}) : \varphi^{-1}(U) \vee x \in \mathfrak{x}\}.$$

Let $\Upsilon = \{\upsilon_U\}_{U \in \mathcal{A}(\mathbf{S})}$. We claim that $(\mathbf{S}, \mathbf{X}, \Upsilon, \Delta)$ is a dual quadruple.

Requirements (1) and (2) of Definition 8.4.1 are satisfied by hypothesis. For (4), let $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathbf{X}$ with $R(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$, so that $\mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. We claim $\Delta(\mathfrak{x}) \bullet \Delta(\mathfrak{y}) \subseteq \Delta(\mathfrak{z})$. To prove this, let $z \in \Delta(\mathfrak{x}) \bullet \Delta(\mathfrak{y})$. Then there exists $x \in \Delta(\mathfrak{x})$ and $y \in \Delta(\mathfrak{y})$ with $x \cdot y \leqslant z$. It follows that $\neg\neg x \in \mathfrak{x}$ and $\neg\neg y \in \mathfrak{y}$, whence $\neg\neg x \cdot \neg\neg y \in \mathfrak{x} \bullet \mathfrak{y} \subseteq \mathfrak{z}$. This yields $\neg\neg(x \cdot y) \in \mathfrak{z}$ as a consequence of $\neg\neg x \cdot \neg\neg y \leqslant \neg\neg(x \cdot y)$, and therefore $\neg\neg z \in \mathfrak{z}$. Thus $z \in \Delta(\mathfrak{z})$, giving (4).

For requirement (3) of Definition 8.4.1, observe that for every $U \in \mathcal{A}(\mathbf{S})$ we have that υ_U is a morphism of GMTL^{τ} as υ_U is the dual of the GMTL-morphism $x \mapsto \varphi^{-1}(U) \vee x$. Define

$$\vee_e(U, W) = \upsilon_U^{-1}[W].$$

for $U \in \mathcal{A}(\mathbf{S})$ and $W \in \mathcal{A}(\mathbf{X})$. We claim that $\vee_e \colon \mathcal{A}(\mathbf{S}) \times \mathcal{A}(\mathbf{X}) \to \mathcal{A}(\mathbf{X})$ gives an external join, viz. that it satisfies Definition 8.1.1(V1,V2,V3).

For (V1), observe that for all $U \in \mathcal{A}(\mathbf{S})$ the map $\vee_e(U, -)$ is an endomorphism of $\mathcal{A}(\mathbf{X})$ by extended Priestley duality. Let $\lambda_W(U) := \vee_e(U, W)$, and let $U, V \in \mathcal{A}(\mathbf{S})$.

Note that $\lambda_W(U) \cup \lambda_W(V) \subseteq \lambda_W(U \cup V)$ follows as a consequence of φ^{-1} being a lattice homomorphism and W being an up-set. Also, application of Lemma 8.2.5(1) yields the reverse inclusion. It follows that $\lambda_W(U) \cup \lambda_W(V) = \lambda_W(U \cup V)$. We may obtain that $\lambda_W(U \cap V) = \lambda_W(U) \cap \lambda_W(V)$ in a similar fashion, which shows that (V1) is satisfied.

For (V2), note that since $\mu_0(\mathfrak{x}) = \mathfrak{x}$ for any $\mathfrak{x} \in \mathbf{X}$ we get that $\upsilon_{\varnothing}^{-1}$ is the identity on $\mathcal{A}(\mathbf{X})$. Moreover, $\upsilon_S^{-1}(W) = W$ for any $W \in \mathcal{A}(\mathbf{X})$ is a consequence of $\mu_1(\mathfrak{x}) = \mathscr{R}(\mathbf{A})$ for any $\mathfrak{x} \in \mathbf{X}$.

To prove (V3), we must show

$$\upsilon_U^{-1}[W] \cup \upsilon_V^{-1}[W'] = \upsilon_{U \cup V}^{-1}[W \cup W'] = \upsilon_U^{-1}[\upsilon_V^{-1}[W \cup W']].$$

One may easily show that

$$\mu_{\varphi^{-1}(U) \cup \varphi^{-1}(V)}(\mathfrak{x}) = \mu_{\varphi^{-1}(V)}(\mu_{\varphi^{-1}(U)}(\mathfrak{x})).$$

This yields $\upsilon_{U \cup V}^{-1}[W \cup W'] = \upsilon_U^{-1}[\upsilon_V^{-1}[W \cup W']]$.

Now let $\mathfrak{x} \in \upsilon_U^{-1}[W] \cup \upsilon_V^{-1}[W']$. Then $\mathfrak{x} \in \upsilon_U^{-1}[W]$ or $\mathfrak{x} \in \upsilon_V^{-1}[W']$, so $\mu_{\varphi^{-1}(U)}(\mathfrak{x}) \in W$ or $\mu_{\varphi^{-1}(V)}(\mathfrak{x}) \in W'$. The sets W and W' are up-sets, so this provides that $\mu_{\varphi^{-1}(U \cup V)}(\mathfrak{x})$ is in each of W, W', so certainly $\upsilon_{U \cup V}(\mathfrak{x}) \in W \cup W'$. Hence $\mathfrak{x} \in \upsilon_{U \cup V}^{-1}[W \cup W']$, giving $\upsilon_U[W] \cup \upsilon_V[W'] \subseteq \upsilon_{U \cup V}[W \cup W']$.

To obtain the last inclusion, let $\mathfrak{x} \in \upsilon_{U \cup V}^{-1}[W \cup W'] = \upsilon_U^{-1}[\upsilon_V^{-1}[W \cup W']]$. Then $\mu_{\varphi^{-1}(U) \cup \varphi^{-1}(V)}(\mathfrak{x}) \in W \cup W'$. Lemma 8.2.5(1) implies that we may assume without loss of generality that

$$\mu_{\varphi^{-1}(U) \cup \varphi^{-1}(V)}(\mathfrak{x}) = \mu_{\varphi^{-1}(U)}(\mathfrak{x}).$$

This gives $\mu_{\varphi^{-1}(U)}(\mathfrak{x}) \in W \cup W'$, and thus $\mu_{\varphi^{-1}(U)}(\mathfrak{x}) \in W$ or $\mu_{\varphi^{-1}(U)}(\mathfrak{x}) \in W'$. That $\mathfrak{x} \in \upsilon_U^{-1}[W] \cup \upsilon_V^{-1}[W']$ follows immediately in the first case, so suppose that $\mu_{\varphi^{-1}(U)}(\mathfrak{x}) \notin W$. From $\mathscr{R}(\mathbf{A}) \in W$, we get $\mu_{\varphi^{-1}(U)}(\mathfrak{x}) \neq \mathscr{R}(\mathbf{A})$ and consequently

$$\mu_{\varphi^{-1}(U)}(\mathfrak{x}) = \mathfrak{x} \in W'$$

by Lemma 8.2.5(4). As W' is an up-set and $\mathfrak{x} \subseteq \mu_{\varphi^{-1}(V)}(\mathfrak{x})$, we obtain $\mu_{\varphi^{-1}(V)}(\mathfrak{x}) \in W'$. It follows that $\mathfrak{x} \in \upsilon_U^{-1}[W] \cup \upsilon_V^{-1}[W']$ in any case, giving (V3) and that $\mathbf{S} \otimes_{\curlyvee}^{\triangle} \mathbf{X}$ is a dual quadruple.

The proof is finished by observing that $\mathbf{S} \otimes_{\curlyvee}^{\triangle} \mathbf{X} \cong \mathcal{S}(\mathbf{A}) \cong \mathbf{Y}$ by the isomorphism $\alpha_{\mathbf{A}}$ defined in Section 8.2 and by the construction of $\mathbf{S} \otimes_{\curlyvee}^{\triangle} \mathbf{X}$. $\square$

Chapter 9

Open problems

The research program developed in the foregoing pages consists of three interlocking components:

1. Duality-theoretic tools for residuated structures, especially those tailored to simplify certain features of particular varieties of interest.

2. Dualized presentations of algebraic constructions on residuated structures, facilitated by and informing the development of the tools alluded to in (1).

3. Purely algebraic analysis of certain varieties of residuated structures and their reducts, aimed both at recasting algebraic structures in a manner amendable to the tools of (1) and discovering aspects of their theory that supports new duality-theoretic results.

We conclude our discussion by offering a few directions for future inquiry in each of these areas.

9.1 Residuated structures

Residuated binars satisfying certain distributive properties (see Section 2.1.1) and normal i-lattices (see Section 2.2) provide the background algebraic theory for the foregoing study. Both of these theories present interesting and difficult open questions, the answers to some of which may implicate duality-theoretic phenomena. The extension of the results we have presented to non-distributive settings is especially relevant.

Question 9.1.1. *What is the relationship between the nontrivial distributive laws* $(\backslash\vee)$, $(\vee/)$, $(\wedge\cdot)$, $(\cdot\wedge)$, $(\wedge\backslash)$, $(/\wedge)$ *in the absence of lattice distributivity?*

The methods used to address the above question in the distributive case are inapplicable in general.

Question 9.1.2. *What can be said of the i-lattice reducts of involutive residuated lattices?*

The duality developed in Chapter 6 answers the above question for Sugihara monoids (albeit very indirectly), but aside from this instance little seems to be known regarding this question.

Question 9.1.3. *What is the quasivariety generated by the forbidden i-lattice* $\mathbf{B}_8$*, and does it admit a useful natural duality?*

Note that the quasivariety generated by the forbidden i-lattice $\mathbf{D}_4$ is the variety of all distributive i-lattices, and its natural duality is a very well known case study (see, e.g., [14]).

9.2 Duality theory for residuated structures

The functional dualities developed in Chapter 4 offered a useful perspective for the dual construction in Chapters 7 and 8. They admit many open questions.

Question 9.2.1. *Do residuation algebras with functional duals generate the variety of all residuation algebras? If not, what is the variety that they generate?*

Question 9.2.2. *Is functionality equivalent to any first-order property of residuation algebras or residuated lattices?*

Although it provides the most generally-applicable framework, extended Priestley duality is often unwieldy in comparison to more tailored duality-theoretic tools (e.g., Esakia duality and the duality for Sugihara monoids given by Chapter 6).

Question 9.2.3. *Are there other simple, Esakia-style dualities for suitably-chosen classes of residuated lattices?*

A residuated lattice $\mathbf{A}$ is called *conic* if each element of A is comparable to the monoid identity of $\mathbf{A}$. Residuated lattices in the variety generated by the conic residuated lattices are called *semiconic*. Due to their proximity to semilinear structures, semiconic residuated lattices seem to be a natural place to look for other well-behaved Esakia-like dualities.

9.3 Dualized constructions

There are many constructions on residuated structures that may admit dualized treatments along the lines of Sugihara monoids and srDL-algebras. Of these, we mention only those for lattice-ordered groups.

Question 9.3.1. *Is there an illuminating dual presentation of the Mundici functor between MV-algebras and lattice-ordered groups with strong order unit?*

Question 9.3.2. *Is there an illuminating dual presentation of the construction of lattice-ordered groups from their negative cones?*

Bibliography

[1] S. Agguzzoli, T. Flaminio, and S. Ugolini. Equivalences between subcategories of MTL-algebras via Boolean algebras and prelinear semihoops. *J. Logic and Computation*, 2017.

[2] A.R. Anderson and N.D. Belnap Jr. *Entailment: The Logic of Relevance and Necessity*, volume 1. Princeton University Press, 1975.

[3] S. Awodey. *Category Theory*, volume 52 of *Oxford Logic Guides*. Oxford University Press, second edition, 2010.

[4] G. Bezhanishvili and S. Ghilardi. An algebraic approach to subframe logics: Intuitionistic case. *Annals of Pure and Applied Logic*, 147:84–100, 2007.

[5] W.J. Blok and D. Pigozzi. Algebraizable logics. *Memoirs of the American Mathematical Society*, 77, 1989.

[6] K. Blount and C. Tsinakis. The structure of residuated lattices. *Internat. J. Algebra Comput.*, 13(4):437–461, 2003.

[7] T.S. Blyth. *Lattices and Ordered Algebraic Structures*. Universitext. Springer-Verlag, London, first edition, 2005.

[8] T.S. Blyth and M.F. Janowitz. *Residuation Theory*, volume 102 of *International Series of Monographs in Pure and Applied Mathematics*. Pergamon, 2014.

[9] S. Burris and H.P. Sankappanavar. *A Course in Universal Algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, 1981.

[10] M. Busaniche, R. Cignoli, and M. Marcos. A categorical equivalence for Stonean residuated lattices. 2017. preprint. Available at https://arxiv.org/abs/1706.06332.

[11] L.M. Cabrer and S.A. Celani. Priestley dualities for some lattice-ordered algebraic structures, including mtl, imtl, and mv-algebras. *Central European Journal of Mathematics*, 4(4):600–623, 2006.

[12] L.M. Cabrer and H.A. Priestley. Gödel algebras: Interactive dualities and their applictions. *Algebra Universalis*, 74:87–116, 2015.

[13] S.A. Celani. Distributive lattices with fusion and implication. *Southest Asian Bulletin of Mathematics*, 28:999–1010, 2004.

[14] D.M. Clark and B.A. Davey. *Natural Dualities for the Working Algebraist*. Cambridge University Press, 1998.

[15] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for non-distributive logics. 2016. Preprint. Available at https://arxiv.org/abs/1603.08515.

[16] J. Czelakowski and D. Pigozzi. Amalgamation and interpolation in abstract algebraic logic. *Algebra Universalis*, 17:34–49, 1983.

[17] B.A. Davey. Dualities for equational classes of brouwerian algebras and heyting algebras. *Trans. Amer. Math. Soc.*, 221:119–146, 1976.

[18] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, New York, second edition, 2002.

[19] B.A. Davey and H. Werner. Dualities and equivalences for varieties of algebras. In A.P. Huhn and E.T. Schmidt, editors, *Contributions to Lattice Theory*, pages 101–275. North-Holland, Amsterdam, New York, 1983.

[20] J.M. Dunn. A Kripke-style semantics for R-mingle using a binary accessibility relation. *Studia Logica*, 35(2):163–172, 1976.

[21] L. Esakia. Topological Kripke models. *Soviet Math. Dokl.*, 15:147–151, 1974.

[22] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124:271–288, 2001.

[23] M.M. Fidel. An algebraic study of a propositional system of Nelson. In N.C. da Costa A.I. Arruda and A.R. Chuaqui, editors, *Proc. of the First Brazilian Conf. on Math. Logic*, pages 99–117. Marcel Dekker, New York, 1978.

[24] W. Fussner and N. Galatos. Categories of models of R-mingle. 2017. Preprint. Available at https://arxiv.org/abs/1710.04256.

[25] W. Fussner and P. Jipsen. Distributive laws in residuated binars. Manuscript, 2018.

[26] W. Fussner and A. Palmigiano. Residuation algebras with functional duals. In Kira Adaricheva et. al., editor, *Algebras and Lattices in Hawai'i: Honoring Ralph Freese, Bill Lampe, and J.B. Nation*, pages 39–46. Lulu, 2018.

[27] W. Fussner and S. Ugolini. A topological approach to MTL-algebras. 2018. Preprint. Available at https://arxiv.org/abs/1808.00841.

[28] N. Galatos. *Varieties of residuated lattices*. PhD thesis, Vanderbilt University, 2003.

[29] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier, 2007.

[30] N. Galatos and J.G. Raftery. A category equivalence for odd Sugihara monoids and its applications. *J. Pure. Appl. Algebra*, 216(10):2177–2192, 2012.

[31] N. Galatos and J.G. Raftery. Idempotent residuated structures: Some category equivalences and their applications. *Trans. Amer. Math. Soc.*, 367(5):3189–3223, 2015.

[32] M. Gehrke. Stone duality, topological algebra, and recognition. *J. Pure Appl. Algebra*, (220):2711–2747, 2016.

[33] M. Gehrke and J. Vosmaer. A view of canonical extensions. In N. Bezhanishvili and L. Spada, editors, *TbiLLC 2009*, LNAI, pages 77–100. Springer, 2011.

[34] S. Givant. *Duality Theories for Boolean Algebras with Operators*. Springer Monographs in Mathematics. Springer, 2014.

[35] R. Goldblatt. Varieties of complex algebras. *Ann. Pure. Appl. Logic*, (44):173–242, 1989.

[36] G. Hansoul. A duality for Boolean algebras with operators. *Algebra Universalis*, 17:34–49, 1983.

[37] R. Jansana and U. Rivieccio. Dualities for modal N4-lattices. *Logic J. of the IGPL*, 22(4):608–637, 2014.

[38] P.T. Johnstone. *Stone Spaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982.

[39] B. Jónsson and A. Tarski. Boolean algebras with operators, part I. *Amer. J. Math.*, 73:891–939, 1951.

[40] B. Jónsson and A. Tarski. Boolean algebras with operators, part II. *Amer. J. Math.*, 74:127–162, 1952.

[41] J.A. Kalman. Lattices with involution. *Trans. Amer. Math. Soc.*, 87(2):485–491, 1958.

[42] M. Kracht. On extensions of intermediate logics by strong negation. *J. Philos. Logic*, 27(1):49–73, 1998.

[43] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, 1998.

[44] C. Noguera. *Algebraic study of axiomatic extensions of triangular norm based fuzzy logics*. PhD thesis, University of Barcelona, 2007.

[45] S.P. Odintsov. Algebraic semantics for paraconsistent Nelson's logic. *J. Logic Comput.*, 13:453–468, 2003.

[46] S.P. Odintsov. On the representation of N4-lattices. *Studia Logica*, 76:385–405, 2004.

[47] S.P. Odintsov. *Constructive Negations and Paraconsistency*. Springer, 2008.

[48] J.S. Olson and J.G. Raftery. Positive Sugihara monoids. *Algebra Univers.*, 57:75–99, 2007.

[49] H.A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, 2(2):186–190, 1970.

[50] H.A. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proc. London Math. Soc.*, 24(3):507–530, 1972.

[51] R. Routley and R.K. Meyer. The semantics of entailment II. *J. Phil. Logic*, 1:53–73, 1972.

[52] R. Routley and R.K. Meyer. The semantics of entailment I. In H. Leblanc, editor, *Truth, Syntax, Modality*. North-Holland Publ. Co., Amsterdam, 1973.

[53] M.H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 4:37–111, 1936.

[54] C. Tsinakis and A.M. Wille. Minimal varieties of involutive residuated lattices. *Studia Logica*, 83(1):407–423, 2006.

[55] S. Ugolini. *Varieties of residuated lattices with an MV-retract and an investigation into state theory*. PhD thesis, Univerity of Pisa, 2018.

[56] A. Urquhart. Duality for algebras of relevant logics. *Studia Logica*, 56(1/2):253–276, 1996.