A Tour of Substructural Interpolation

Wesley Fussner

Institute of Computer Science
Czech Academy of Sciences

Past and on-going work, both individual and joint with various subsets of
{Nick Galatos, Peter Jipsen, George Metcalfe, Simon Santschi}

Seminar on Applied Mathematical Logic
Prague, Czechia

8 November 2023

Interpolation: The Idea

Interpolation refers to a cluster of metalogical properties asserting that if A entails B, then there exists I dealing only with the subject matter common to A and B and such that

$$A \text{ entails } I \text{ and } I \text{ entails } B.$$

I is called an interpolant and gives a kind of explanation for why A entails B.

The intuitive notions 'entailment' and 'common subject matter' are cashed out in many different ways.

Interpolation: Some Variants

Craig interpolation:

$$A \to B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), A \to I, I \to B)$$

Deductive interpolation:

$$A \vdash B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), A \vdash I, I \vdash B)$$

Uniform interpolation: For any $S \subseteq \mathrm{var}(A)$, there exists I, $\mathrm{var}(I) \subseteq S$, so that if $\mathrm{var}(A) \cap \mathrm{var}(B) \subseteq S$,

$$A \to B \iff I \to B$$

Interpolation: Applications

- Philosophical issues (e.g. argumentation theory)
- Hardware and software verification (Ken McMillan approach)
- Database theory (primarily uniform interpolation)
- Recent efforts to use interpolants as a resource for SMT solving.

Interpolation: A Short Bibliography

- W. Fussner and N. Galatos, Semiconic idempotent logic I: Structure and local deduction theorems. https://arxiv.org/abs/2208.09724v2.
- W. Fussner and N. Galatos, Semiconic idempotent logic II: Beth definability and deductive interpolation. https://arxiv.org/abs/2208.09724v3.
- W. Fussner and G. Metcalfe, Transfer theorems for finitely subdirectly irreducible algebras. To appear in *Journal of Algebra*. https://arxiv.org/abs/2205.05148.
- W. Fussner, G. Metcalfe, and S. Santschi, Interpolation and the Exchange rule. https://arxiv.org/abs/2310.14953
- W. Fussner and S. Santschi, Interpolation in Linear Logic and Related Systems. https://arxiv.org/abs/2305.05051

Part I:
Tools and Techniques

- Probably any discussion of interpolation in substructural logics should begin with Maehara's method.
- Available when a suitable analytic sequent calculus (or sometimes tableau system) is available.
- Essentially operates by iteratively partitioning pairs of sets of formulas in two parts, with a variable in common.
- When successful, results in a (constructively produced!) Craig interpolant.
- .... But sharply limited by the existence of a good proof theory.

Interpolation via Amalgamation

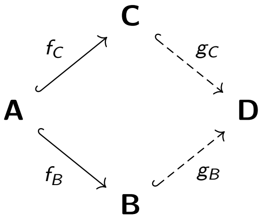
Theorem (Czelakowski-Pigozzi 1999):

Let $\vdash$ be an algebraizable deductive system with a local deduction theorem whose equivalent algebraic semantics is the variety $\mathcal{V}$. Then $\vdash$ has the deductive interpolation property if and only if $\mathcal{V}$ has the amalgamation property.

Amalgamation

Definition:

Let K be a class of algebraic structures. A span in K is a quintuple (A, B, C, f_B, f_C), where $A, B, C \in$ K and $f_B \colon A \to B$, $f_C \colon A \to C$ are embeddings. We say that K has the amalgamation property (or AP) if for every span (A, B, C, f_B, f_C) in K there exists $D \in$ K and embeddings $g_B \colon B \to D$ and $g_C \colon C \to D$ such that $g_B \circ f_B = g_C \circ f_C$.



- Quasivariety: Class defined by quasiequations, or, equivalently, closed under isomorphisms,subalgebras, direct products, and ultraproducts.
- If Q is a quasivariety and $\mathbf{A} \in \mathsf{Q}$, a congruence Θ of $\mathbf{A}$ is a Q-congruence if $\mathbf{A}/\Theta \in \mathsf{Q}$.
- $\mathbf{A}$ is finitely Q-subdirectly irreducible if the least congruence Δ is meet-irreducible in $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$.
- If Q is clear, we just call these relatively finitely subdirectly irreducible and denote the class of them by $\mathsf{Q}_{\mathsf{RFSI}}$.
- Q is Q-congruence distributive if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ is distributive.
- Q has the Q-congruence extension property for each $\mathbf{B} \in \mathsf{Q}$, if $\mathbf{A} \leq \mathbf{B}$ and $\Theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$, then there exists $\Psi \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{B})$ such that $\Theta = \Psi \cap A^2$.

Major question: How to establish/refute the AP.
Answer: Reduce the complexity of the problem to some tractable generating class.

Theorem (F.-Metcalfe 2023):

Let K be a subclass of a quasivariety Q satisfying

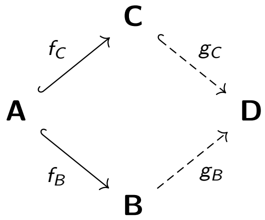
1. K is closed under isomorphisms and subalgebras;
2. every relatively subdirectly irreducible member of Q belongs to K;
3. for any $\mathbf{B} \in \mathsf{Q}$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if $\Theta \in \mathrm{Con}_{\mathsf{Q}} \mathbf{A}$ and $\mathbf{A}/\Theta \in \mathsf{K}$, then there exists a $\Phi \in \mathrm{Con}_{\mathsf{Q}} \mathbf{B}$ such that $\Phi \cap A^2 = \Theta$ and $\mathbf{B}/\Phi \in \mathsf{K}$;
4. every doubly injective span of finitely generated algebras in K has an amalgam in Q.

Then Q has the amalgamation property.

One-Sided Amalgamation

Definition:

Let K be a class of algebraic structures. We say that K has the one-sided amalgamation property (or 1AP) if for every span (A, B, C, f_B, f_C) in K there exists $D \in K$, an embedding $g_B\colon B \to D$, and a homomorphism $g_C\colon C \to D$ such that $g_B \circ f_B = g_C \circ f_C$.



Characterizing the AP

Theorem (F.-Metcalfe 2023):

Let Q be any quasivariety with the Q-congruence extension property such that Q_{RFSI} is closed under subalgebras. The following are equivalent:

1. Q has the amalgamation property.
2. Q has the one-sided amalgamation property.
3. Q_{RFSI} has the one-sided amalgamation property.
4. Every doubly injective span in Q_{RFSI} has an amalgam in $Q_{RFSI} \times Q_{RFSI}$.
5. Every doubly injective span of finitely generated algebras in Q_{RFSI} has an amalgam in Q.

Simply Tabular Logics

Many results follow promptly from the previous characterization. For example, the following is useful in studying various kinds of 'tabular' logics.

Definition:

We say that an algebra $\mathbf{A}$ is extensible if whenever $\mathbf{B}, \mathbf{B}'$ are subalgebras of $\mathbf{A}$ and $h\colon \mathbf{B} \to \mathbf{B}'$ is an isomorphism, there exists an automorphism $\hat{h}\colon \mathbf{A} \to \mathbf{A}$ extending h.

Theorem (F. 2023+):

Let $\mathbf{A}$ be a finite simple algebra with CEP, and assume that $\mathsf{V} = \mathsf{V}(\mathbf{A})$ is congruence-distributive. Further assume that $\mathbf{A}$ has no 1-element subalgebras. Then V has the amalgamation property if and only if $\mathbf{A}$ is extensible.

Decidability

Definition:

A variety finitely generated if it is generated as a variety by some given finite set of finite algebras of finite signature.

Theorem (F.-Metcalfe 2023):

Let V be a finitely generated congruence-distributive variety such that V_{FSI} is closed under subalgebras. There exist effective algorithms to decide if V has the congruence extension property and amalgamation property.

Heuristic Approaches

The complexity of the decision procedure is quite bad and we often want to use computational resources for varieties that aren't finitely generated. In these cases, some heuristic approaches are available. Basic recipe:

1. Enumerate small triples $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ with $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{A} \leq \mathbf{C}$.
2. Write down the atomic diagrams of $\mathbf{B}$, $\mathbf{C}$ with appropriate names for $A = B \cap C$.
3. Feed the resulting constraints into a prover/model builder.
4. Simultaneously run a search for models ($=$ amalgamable span) and proofs of falsum ($=$ proof that the span cannot be amalgamated).

This is a necessarily interactive approach: Need to know something about the algebras to optimize computation.

Injectives

Definition:

Let K be a class of similar algebras. An algebra $\mathbf{Q} \in \mathsf{K}$ is called injective over K if for all algebras $\mathbf{A}, \mathbf{B} \in \mathsf{K}$, embedding $\alpha \colon \mathbf{B} \to \mathbf{A}$, and homomorphism $\beta \colon \mathbf{B} \to \mathbf{Q}$ there exists a homomorphism $\phi \colon \mathbf{A} \to \mathbf{Q}$ such that $\phi \circ \alpha = \beta$.

The class K is said to have enough injectives if every algebra in K embeds into an algebra in K that is injective over K.

Theorem (??? Kiss et al. 1983):

Let K be a class of similar algebras that is closed under taking finite products. If K has enough injectives, then it has the amalgamation property.

Part II:
What We Know

Some Milestones

- Interpolation for propositional and first-order classical logic (Craig 1957).
- Exactly 8 superintuitionistic logics with Craig/deductive interpolation (Maksimova 1977); uniform interpolation for all of these (Pitts 1992, Ghilardi and Zawadowski 2002).
- Failure of DIP + CIP for many relevance logics (Urquhart 1993); classification of DIP + CIP for relevance logics with mingle (Marchioni and Metcalfe 2012).
- Classification of DIP for Łukasiewicz logic (Di Nola and Lettieri 2000).
- Uncountably many extensions of Hájek's basic fuzzy logic without DIP (Montagna 2006); lots of positive partial results (Aguzzoli and Bianchi 2021, 2023).

Residuated Lattices

A residuated lattice is an algebraic structure of the form $(A, \wedge, \vee, \cdot, \backslash, /, e)$ where

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, e)$ is a monoid, and
- for all $x, y, z \in A$,

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y.$$

We use all the expected terminology: Commutative, idempotent, totally ordered, linear, etc.

Semilinear: Subalgebra of a direct product of totally ordered residuated lattices.
Pointed: Additional constant f (usually called FL-algebras).

Basic Structural Rules

$$\frac{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_2, \Pi_1, \Gamma_2 \Rightarrow \Delta} \text{ (e)} \qquad \frac{\Gamma_1, \Pi, \Pi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta} \text{ (c)}$$

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta} \text{ (i)} \qquad \frac{\Pi \Rightarrow}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta} \text{ (o)}$$

$$\frac{\Gamma_1, \Pi_1, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta} \text{ (m)}$$

Exchange: A Critical Rule

State of the art up until a few years ago:

- Lots of success finding logics with exchange and DIP, but not much systemic issue.
- Much worse without exchange: Previously thought that there may be no extension of FL lacking exchange with DIP (Gil Férez-Ledda-Tsinakis 2015).
- Example given in 2020 by Gil-Férez, Jipsen, Metcalfe.
- Several natural examples involving the law of excluded middle (F.-Galatos 2022).

Logics with Exchange

Theorem (F.-Santschi 2023+):

- Each of the varieties of commutative residuated lattices and pointed commutative residuated lattices has continuum-many subvarieties with the AP.
- Consequently, each of FL_e and its falsum-free fragment have continuum-many axiomatic extensions with the DIP.

Theorem (F.-Santschi 2023+):

- Each of the varieties of commutative residuated lattices and pointed commutative residuated lattices has continuum-many subvarieties without the AP.
- Consequently, each of FL_e and its falsum-free fragment have continuum-many axiomatic extensions without the DIP, hence without CIP.

- Actually, the previous results can be extended by adding an involution, bounds, and the linear logic modalities (! and ?).
- Here for the cases with DIP, we also get a restricted form of Craig interpolation:

$$!A \to !B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), !A \to !I, !I \to !B)$$

- Proof relies on deep properties of abelian groups plus some constructions from algebraic linear logic.

Logics Without Exchange

Theorem (F.-Metcalfe-Santschi 2023+):

1. There are continuum-many varieties of idempotent semilinear residuated lattices that have the amalgamation property and contain non-commutative members.
2. There are continuum-many axiomatic extensions of $\mathsf{SemRL}_{\mathsf{cm}}$ that have the deductive interpolation property in which the exchange rule is not derivable.

Theorem (F.-Metcalfe-Santschi 2023+):

1. There are exactly sixty varieties of commutative idempotent semilinear residuated lattices that have the amalgamation property.
2. There are exactly sixty axiomatic extensions of $\mathsf{SemRL}_{\mathsf{ecm}}$ that have the deductive interpolation property.

More on Logics Without Exchange

- The previous results concern variants of substructural logics without the constant f (i.e., residuated lattices rather than pointed residuated lattices).
- In the case with f, we can still show that there are finitely many varieties with amalgamation, but there are many more ($> 12,500,000$).
- The proof uses totally different methods than the commutative case. Here we have to focus on the dynamics of bi-infinite words to construct the logics with DIP.

Part III:
What We'd Like to Know

Charting the Basic Structural Rules

Question:

How many axiomatic extensions of FL_{ecm} (no semilinearity) have DIP/CIP?

Question:

Are there continuum-many logics with weakening that have DIP? CIP? What about if we add exchange?

Question:

Does the picture change with knotted extensions?

Computation and finiteness

Question:

Let V be a congruence distributive variety whose class of finitely subdirectly irreducibles is closed under subalgebras. If V is finitely presented, is it decidable whether V has CEP, AP, and so forth?

Extending this work to finitely presented algebras is especially interesting because of its connection to uniform interpolation: A variety V has right uniform deductive interpolation if and only it has DIP and is coherent (any finitely generated subalgebra of a finitely presented algebra is finitely presented).

New Transfer Theorems

Question:

Can we develop good transfer theorems for studying Craig interpolation, uniform interpolation, and so on?

Question:

Can we obtain more convenient transfer theorems for studying quasivarieties, in particular in terms of critical algebras?

Thank you!