# The structure of totally ordered idempotent residuated lattices

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# Sketch of the talk

- Residuated lattices: Equivalent algebraic semantics for substructural logics.
- Extensively studied both from logical perspective and from classical algebra, as well as connections to algebraic proof theory.
- Idempotence: Important both as building blocks for other residuated structures, and complements studies of structures in the cancellative family ($\ell$-groups, MV-algebras, ...).
- This talk: Describe structure on three levels—ordinal sums of totally ordered posets, formulation as idempotent Galois connections, and enhanced monoidal preorders.
- Applications to semilinear varieties: Congruence extension property, strong amalgamation property, epimorphism surjectivity, and corresponding logical properties.

# Residuated lattices

A residuated lattice is an algebraic structure of the form $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ where

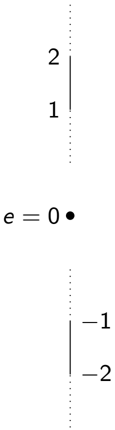
- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a monoid, and
- for all $x, y, z \in A$,

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y.$$

**Familiar examples:** Boolean algebras, Heyting algebras, MV-algebras, relation algebras, lattice-ordered groups, algebras of ideals of rings, etc.

**This talk:** $(A, \wedge, \vee)$ is totally ordered and $x \cdot x = x$.

# Idempotent chains: The Sugihara monoids



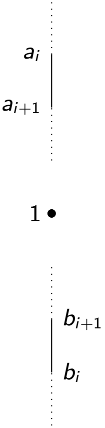

Odd Sugihara monoids are commutative, idempotent, distributive residuated lattices where the map $x \mapsto x\backslash 0 = 0/x$ $(= x \to 0)$ is an involution. They are generated by a generic example given on the integers $\mathbb{Z}$.

Multiplication is the meet with respect to a non-standard order:

$$\ldots < -2 < 2 < -1 < 1 < 0 = e$$

This means $x \cdot y$ is whichever of $x$ or $y$ is further away from the identity, unless there is a tie (then it is the meet).

# Idempotent chains: The non-commutative case




In 2004, Galatos gave lots of generalizations to the non-commutative case built on an index set ($\mathbb{Z}$ or $\mathbb{N}$ or a finite set). Main ingredient is a subset $J$ of the index set that determines how to break ties.

The product of two elements is the furthest away from 1, e.g. $b_0 a_1 = a_1 b_0 = b_0$. If there is a tie, then:

- if $i \in J$, then $a_i b_i = a_i$ and $b_i a_i = b_i$ (Left), and
- if $i \notin J$, then $a_i b_i = b_i$ and $b_i a_i = a_i$ (Right).

The inverse operations $x^\ell := 1/x$ and $x^r := x\backslash 1$ play an important role.

# Inverses as a nuclear image

Recall that a nucleus on a residuated lattice is a closure operator $\gamma$ that satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(xy)$.

For an idempotent residuated chain $\mathbf{A}$, we define the map $\gamma$ on $A$ by

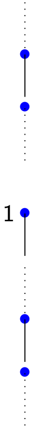
$$\gamma(x) = x^{\ell r} \wedge x^{r\ell}.$$

**Lemma:**

If $\mathbf{A}$ is an idempotent residuated chain, then

1. $\gamma$ is a nucleus.
2. $\gamma[A] := A^i = \{x^\ell : x \in A\} \cup \{x^r : x \in A\}$.
3. $\gamma(x)^r = x^r$ and $\gamma(x)^\ell = x^\ell$, for all $x \in A$.
4. The sets $\gamma^{-1}(a)$, where $a \in A^i$, form convex subposets of $\mathbf{A}$ with top element $a$, and they are ordered linearly according to the value of $a$: if $x \in \gamma^{-1}(a)$ and $y \in \gamma^{-1}(b)$, where $a, b$ are distinct elements of $A^i$, then $x \leq y$ iff $a \leq b$.

# Quasi-involutivity




**Definition:**

An idempotent residuated chain is called quasi-involutive if it satisfies $x = x^{\ell r} \wedge x^{r\ell}$.

- The set of inverses $A^i$ is the universe of a subalgebra of $\mathbf{A}$.
- $\mathbf{A}^i$ is quasi-involutive.
- If $\mathbf{A}$ is commutative, then $x^r = x^\ell$ and $\mathbf{A}^i$ is an odd Sugihara monoid.
- We'll often call $\mathbf{A}^i$ the skeleton of $\mathbf{A}$.

# Operations from the skeleton

Actually, all operations of an idempotent residuated chain can be completely described terms of the skeleton.

**Lemma:**

In every idempotent residuated chain we have

$$xy = \begin{cases} x & y \in (x^r, x] \text{ or } y \in [x, x^r] \\ y & x \in (y^\ell, y] \text{ or } x \in [y, y^\ell] \end{cases}$$

$$x \backslash y = \begin{cases} x^r \vee y & x \leq y \\ x^r \wedge y & y < x \end{cases} \qquad y/x = \begin{cases} x^\ell \vee y & x \leq y \\ x^\ell \wedge y & y < x \end{cases}$$

# Inverses and centrality

We call an element $a$ central if $ax = xa$ for all $x$, and write $C(a) := \{x : ax = xa\}$. Define also $x^* := x^\ell \vee x^r$, and recall that a semigroup $\mathbf{S}$ is left-zero if it satisfies $xy = x$ and right-zero it satisfies $xy = y$.

**Lemma:**

Let $a$ be an element of an idempotent residuated chain $\mathbf{A}$. Then:

1. If $a$ is central (equivalenty, $a^\ell = a^r$), then $C(a) = A$ and $\{a, a^*\}$ forms a semilattice with multiplication equal to the inherited meet.
2. If $a$ is not central (equivalenty, $a^\ell \neq a^r$), then $C(a) = \{a^*\}^c$ and $\{a, a^*\}$ forms a left-zero or a right-zero semigroup.

Also, there is no element between the elements $a^\ell$ and $a^r$ and these elements form a covering pair.

# Level 1: Ordinal sums

**Definition:**

A decomposition system is a pair $(\mathbf{S}, \{\mathbf{A}_s : s \in S\})$, where

1. $\mathbf{S}$ is a quasi-involutive idempotent residuated chain.
2. For all $s \in S$, $\mathbf{A}_s$ is a totally ordered poset with top element $s$.
3. If $s$ is not central, then $A_s$ is trivial.

For each decomposition system, you can define an idempotent residuated chain based on the ordinal sum $\bigoplus_{s \in S} \mathbf{A}_s$ where the operations are as before.

**Theorem:**

Every idempotent residuated chain is isomorphic to the ordinal sum corresponding to its decomposition system.

**Caution:** This is the ordinal sum of posets/lattices, not as in e.g. BL-algebras.

# A splitting

Recall that $x^* = x^\ell \vee x^r$, and further define $x^\star = x^\ell \wedge x^r$.

**Lemma:**

For all $a, b$ in a residuated residuated lattice we have $a \leq b^\star$ iff $b \leq a^\star$. In other words the pair $(a^\star, a^\star)$ is a Galois connection. Moreover, in idempotent residuated chains, $(a^\star, a^*)$ forms a splitting pair: for all $c$, $c \leq a^\star$ or $a^* \leq c$.

# Level 2: Idempotent Galois connections

**Definition:**

An idempotent Galois connection is an algebra $(A, \wedge, \vee, {}^\ell, {}^r, 1)$ such that:

1. $(A, \wedge, \vee)$ is a chain.
2. $({}^\ell, {}^r)$ forms a Galois connection.
3. $1^\ell = 1^r = 1$.
4. For all $x$, there is no element between $x^\ell$ and $x^r$ (i.e., $(x^\star, x^*)$ forms a splitting pair).

The $\{\wedge, \vee, {}^\ell, {}^r, 1\}$-reduct of an idempotent residuated chain forms an idempotent Galois connection, and in fact...

**Theorem:**

Idempotent residuated chains are definitionally-equivalent to idempotent Galois connections.

# The natural order and the monoidal order

Previous work on idempotent residuated chains (e.g. GJM20, CZ09) have used two ordering relations connected to the product operation:

- The natural order: $x \leq_n y$ iff $xy = yx = x$.
- The monoidal preorder: $x \sqsubseteq y$ iff $xy = x$.

Abbreviate $x \sqsubset y$ iff $x \sqsubseteq y$ and $y \not\sqsubseteq x$.

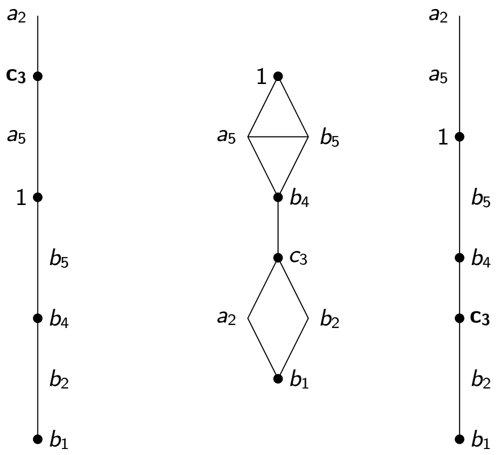
**Lemma:**

The following hold in idempotent residuated chains.

1. The relation $\leq_n$ is an order the relation $\sqsubseteq$ is a preorder.
2. $x <_n y$ iff $x \sqsubset y$, for all $x, y$.
3. $xy = x$ iff $x \sqsubseteq y$. Also, $xy = y$ iff $y \not\sqsubseteq x$.
4. $x \sqsubset y$ iff $y \neq x = xy = yx$.

# Insufficiency of the monoidal preorder

The monoidal preorder contains more information than the natural order, but it isn't enough to capture the structure of idempotent residuated chains:



# Level 3: Enhanced monoidal preorders

We can fix this problem with the monoidal preorder by enriching the structure: We add information about which elements are positive and which are negative.

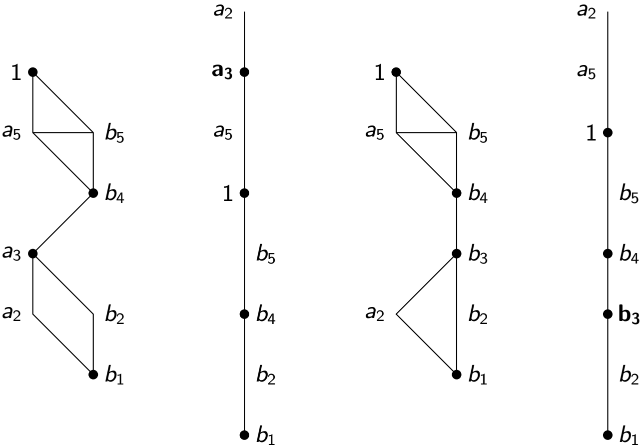
**Definition:**

$(P, \sqsubseteq, P^+, P^-, 1, ^\star)$ is an enhanced monoidal preorder if: $(P, \sqsubseteq)$ is a pre-ordered set with sole maximum element 1 ($x \sqsubset 1$, for all $x \neq 1$), $P^+$ and $P^-$ are totally ordered subsets of $P$ (i.e., the restriction of $\sqsubseteq$ to each of $P^+$, $P^-$ antisymmetric and total) with $P^+ \cup P^- = P$ and $P^+ \cap P^- = \{1\}$, and $^\star$ is a unary operation on $P$ such that $1^\star = 1$ and for all other elements

1. for $b \in P^-$, $b^\star$ is the smallest element of $P^+$ such that $b \sqsubset b^\star$
2. for $a \in P^+$, $a^\star$ is the largest element of $P^-$ such that $a^\star \sqsubset a$.
3. the preordered is *layered*: if two distinct elements are not related by $\sqsubset$ nor $\sqsupset$, then they have different signs and their $\sqsubset$-upsets and downsets coincide.

# Level 3: Enhanced monoidal preorders

In enhanced monoidal preorders, positive elements are drawn on the left and negative elements on the right . We can distinguish idempotent residuated chains with the same monoidal preorder under this regime:



# Level 3: Enhanced monoidal preorders

For an idempotent residuated chain, set $A^{+} = \{a \in A : 1 \leq a\}$ and $A^{-} = \{a \in A : a \leq 1\}$. Then $(A, \sqsubseteq, A^{+}, A^{-}, 1, {}^{\star})$ is an enhanced monoidal preorder. In fact:

**Theorem:**

Idempotent residuated chains are definitionally-equivalent to enhanced monoidal preorders.

We will sketch the inverse of the correspondence: Building idempotent residuated chains from enhanced monoidal preorders.

# From enhanced monoidal preorders to residuated chains

Given an enhanced monoidal preorder $(P, \sqsubseteq, P^+, P^-, 1, {}^\star)$, define the ordered algebra $\mathbf{A}$ with underlying set $A = P$, with order given by:

$$x \leq y \text{ iff } (x, y \in P^- \text{ and } x \sqsubseteq y) \text{ or } (x, y \in P^+ \text{ and } y \sqsubseteq x) \text{ or } (x \in P^- \text{ and } y \in P^+)$$

with inverses by $x^\ell = x^r = x^\star$, if $x$ is a $\sqsubseteq$-conical element ($\sqsubseteq$-comparable to every element), and for $a \in P^+$ and $b \in P^-$

1. $a^\ell = b$, $a^r = a^\star$, $b^\ell = b^\star$, $b^r = a$, if $a, b$ are mutually comparable and
2. $a^\ell = a^\star$, $a^r = b$, $b^\ell = a$, $b^r = b^\star$, if $a, b$ are incomparable

This gives an idempotent Galois connection that can be converted into an idempotent residuated chain.

# An application: Amalgamation

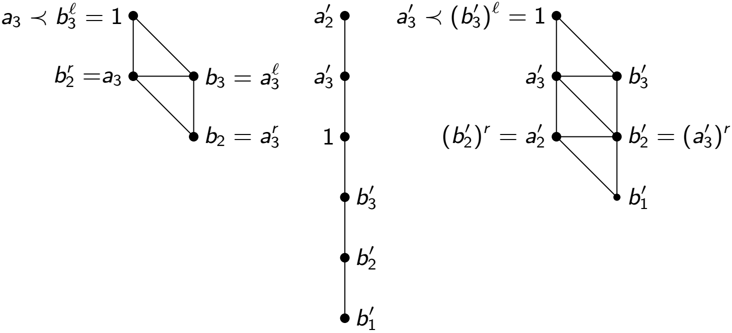
Intended application of all of this was to obtain the strong amalgamation property for semilinear residuated lattices.

A span in a class $\mathcal{K}$ of algebras is a 5-tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f_{\mathbf{B}}, f_{\mathbf{C}})$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $f_{\mathbf{B}} \colon \mathbf{A} \to \mathbf{B}$ and $f_{\mathbf{C}} \colon \mathbf{A} \to \mathbf{C}$ are embeddings.

- $\mathcal{K}$ has the amalgamation property if for every span $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f_{\mathbf{B}}, f_{\mathbf{C}})$ in $\mathcal{K}$ there exists a triple $(\mathbf{D}, g_{\mathbf{B}}, g_{\mathbf{C}})$ so that $g_{\mathbf{B}} \colon \mathbf{B} \to \mathbf{D}$ and $g_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{D}$ are embdeddings and $g_{\mathbf{B}} f_{\mathbf{B}} = g_{\mathbf{C}} f_{\mathbf{C}}$ (i.e., the corresponding diagram commutes).
- $\mathcal{K}$ has the strong amalgamation property if the amalgam above can be taken so that the intersection of the images of $g_{\mathbf{B}}$ and $g_{\mathbf{C}}$ is exactly the image of $A$.

# Amalgamation: Failure

It turns out that the amalgamation property fails for idempotent residuated chains (and even semilinear idempotent RLs):



# $^\star$-involutivity

Recall that $x^\star = x^\ell \wedge x^r$. An idempotent residuated chain is called $^\star$-involutive if it satisfies $x^{\star\star} = x$.

**Theorem:**

The class of $^\star$-involutive idempotent residuated chains has the strong amalgamation property.

**Proof idea:** Rests on a characterization of the enhanced monoidal preorders of $^\star$-involutive idempotent residuated chains. Roughly the amalgam of $\mathbf{A}$ and $\mathbf{B}$ is constructed as an ordinal sum (corresponding to a nested sum) of their enhanced monoidal preorders.

# Lifting amalgamation

Using some general universal algebraic results, we can lift the amalgamation property from the class of $^{\star}$-involutive residuated chains to the variety they generate:

**Theorem:**

The variety of $^{\star}$-involutive semilinear idempotent residuated lattices has the strong amalgamation property.

To do this, we need to show that the class has the congruence extension property.

# The congruence extension property

An algebra $\mathbf{A}$ has the congruence extension property if whenever $\mathbf{B} \leq \mathbf{A}$ and $\Theta$ is a congruence on $\mathbf{B}$, there exists a congruence $\Psi$ on $\mathbf{A}$ such that $\Theta = \Psi \cap A^2$.

**Theorem**

The variety of semilinear idempotent residuated lattices has the congruence extension property.

**Proof (sketch):** Key idea is that congruences of residuated lattices are given by filters that are closed under iterated conjugates, i.e., compositions of maps of the form $a \mapsto (xa)/x \wedge 1$ and $a \mapsto (x\backslash ax) \wedge 1$. One can use the structural results we've given to show that semilinear idempotent residuated lattices satisfy $y \wedge y^{\ell\ell} \wedge y^{rr} \leq xy/x, x\backslash yx$, and this allows us to eliminate iterated conjugates.

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# Thank you!