

# Interpolation via Finitely Subdirectly Irreducible Algebras

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# Metalogical properties and algebra

This talk is about **practical algebraic tools** to study **metalogical properties**. In particular, if  $\vdash$  is a deductive system we say:

- $\vdash$  has **the deductive interpolation property** if for all formulas  $\varphi, \psi$  such that  $\varphi \vdash \psi$ , there exists a formula  $\sigma$  such that  $\varphi \vdash \sigma$ ,  $\sigma \vdash \psi$ , and the variables of  $\sigma$  are among those appearing in both of  $\varphi$  and  $\psi$ .
- $\vdash$  has a **local deduction theorem** if there exists a family  $\{d_j(p, q) : j \in J\}$  of sets of formulas  $d_j(p, q)$  in at most two variables such that for every set of formula  $\Gamma \cup \{\varphi, \psi\}$  we have

$$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash d_j(\varphi, \psi) \text{ for some } j \in J.$$

(Think  $d_j(\varphi, \psi) = \{\varphi \rightarrow \psi\}$ ).

When  $\vdash$  is **algebraizable**, these properties correspond to the **the amalgamation property** and the **the congruence extension property**.

# The breadth of application

- Most everyday logics are algebraizable (see Blok-Pigozzi 1989):
  - Intuitionistic logic, normal modal logics, multiplicative-additive linear logic, relevance logics above  $\mathbf{R}$ , Łukasiewicz logic, t-norm based fuzzy logics, classical first-order logic, ...
- This study applies in a language-agnostic way (may not have  $\rightarrow$ , can't express Craig interpolation).
  - Consider fragments of well-known logics.
- Applies even when there is no good proof theory.
  - Challenging extensions of Full Lambek calculus like GBL, Łukasiewicz, ...
- Main limitation is set by whether it is difficult to analyze the pertinent algebraic models.

# Some basic definitions

- A **variety** is a class  $\mathcal{V}$  of algebras defined by equations, or equivalently, **closed under homomorphic images, direct products, and subalgebras**.
- An algebra  $\mathbf{A}$  is called **subdirectly irreducible** if whenever  $\mathbf{A}$  is isomorphic to a subdirect product of a set of algebras, it is isomorphic to one of these algebras. Equivalently,  $\mathbf{A}$  is subdirectly irreducible if the least congruence  $\Delta_{\mathbf{A}} = \{\langle a, a \rangle : a \in A\}$  is **completely meet-irreducible** in the lattice of congruences  $\text{Con } \mathbf{A}$  of  $\mathbf{A}$ . We denote the subdirectly irreducibles of a variety  $\mathcal{V}$  by  $\mathcal{V}_{\text{SI}}$ .
- An algebra  $\mathbf{A}$  is called **finitely subdirectly irreducible** if whenever  $\mathbf{A}$  is isomorphic to a subdirect product of a non-empty, finite set of algebras, it is isomorphic to one of these algebras. Equivalently,  $\mathbf{A}$  is finitely subdirectly irreducible if  $\Delta_{\mathbf{A}}$  is **meet-irreducible** in  $\text{Con } \mathbf{A}$ . We denote the finitely subdirectly irreducibles of a variety  $\mathcal{V}$  by  $\mathcal{V}_{\text{FSI}}$ .

# From SI algebras to FSI algebras

- Establishing that a variety  $\mathcal{V}$  has some property by arguing on  $\mathcal{V}_{SI}$  is a common proof strategy in universal algebra.
- Theme of today's talk: One often obtains **simpler, more elegant, and more useful** formulations of transfer theorems (from  $\mathcal{V}_{FSI}$  to  $\mathcal{V}$ ) when working with FSI algebras instead of SI algebras, especially for logically-relevant properties and varieties.
- One reason: Varieties corresponding to logics often have **equationally definable principal meets**, which implies  $\mathcal{V}_{FSI}$  is a universal class.
- Often obtain **equivalent formulations** of properties in terms of  $\mathcal{V}_{FSI}$  that do not exist for  $\mathcal{V}_{SI}$ .

# The congruence extension property

## Definition:

An algebra  $\mathbf{A}$  is **congruence-distributive** if  $\text{Con } \mathbf{A}$  is a distributive lattice.  $\mathbf{A}$  has the **congruence extension property** (or CEP) if for any subalgebra  $\mathbf{A}$  of  $\mathbf{B}$  and  $\Theta \in \text{Con } \mathbf{A}$ , there exists a  $\Phi \in \text{Con } \mathbf{B}$  such that  $\Phi \cap A^2 = \Theta$ . A class of algebras  $\mathcal{K}$  has these properties if each  $\mathbf{A} \in \mathcal{K}$  does.

Our first illustration of FSI algebras in action:

## Theorem:

Let  $\mathcal{V}$  be any congruence-distributive variety. Then  $\mathcal{V}$  has the congruence extension property if and only if  $\mathcal{V}_{\text{FSI}}$  has the congruence extension property.

# Comparison to SI algebras

The previous theorem improves on the following similar results for SIs, which is also a corollary by noting that each member of  $\mathcal{V}_{\text{FSI}}$  embeds into an **ultraproduct** of members of  $\mathcal{V}_{\text{SI}}$ :

## Corollary (Davey 1977):

Let  $\mathcal{V}$  be any congruence-distributive variety such that  $\mathcal{V}_{\text{SI}}$  is an elementary class. Then  $\mathcal{V}$  has the congruence extension property if and only if  $\mathcal{V}_{\text{SI}}$  has the congruence extension property.

Major improvement is that  $\mathcal{V}_{\text{SI}}$  being an elementary class is a **very strong hypothesis**, and not needed for  $\mathcal{V}_{\text{FSI}}$ .

Often the CEP is reformulated using **commutative diagrams**.

## Definition:

Let  $\mathcal{K}$  be a class of similar algebras. A **span** in  $\mathcal{K}$  is a 5-tuple  $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$  consisting of  $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$  and homomorphisms  $\varphi_B: \mathbf{A} \rightarrow \mathbf{B}$ ,  $\varphi_C: \mathbf{A} \rightarrow \mathbf{C}$ . A span is:

- **injective** if  $\varphi_B$  is an embedding.
- **doubly injective** if both  $\varphi_B$  and  $\varphi_C$  are embeddings.
- **injective-surjective** if  $\varphi_B$  is an embedding and  $\varphi_C$  is surjective.



# The extension property

## Definition:

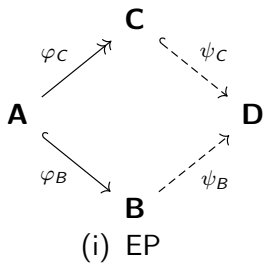
A class  $\mathcal{K}$  of similar algebras has the **extension property** (or EP) if for any injective-surjective span  $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$  in  $\mathcal{K}$ , there exist a  $\mathbf{D} \in \mathcal{K}$ , a homomorphism  $\psi_B: \mathbf{B} \rightarrow \mathbf{D}$ , and an embedding  $\psi_C: \mathbf{C} \rightarrow \mathbf{D}$  such that  $\psi_B \varphi_B = \psi_C \varphi_C$ .

## Proposition (Bacsich 1972):

A variety  $\mathcal{V}$  has the CEP if and only if it has the EP.

**Note:** This **need not be true** for other classes of algebras.

# The extension property



# A transfer theorem for the EP

Our previous characterization of the CEP in terms of  $\mathcal{V}_{\text{FSI}}$  does not make any special demands on  $\mathcal{V}_{\text{FSI}}$  itself, but with additional hypotheses we can say more:

## Theorem:

Let  $\mathcal{V}$  be a congruence-distributive variety such that  $\mathcal{V}_{\text{FSI}}$  is **closed under subalgebras**. The following are equivalent:

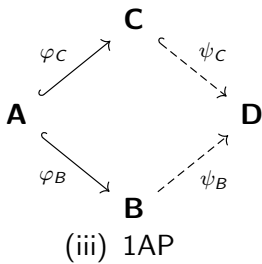
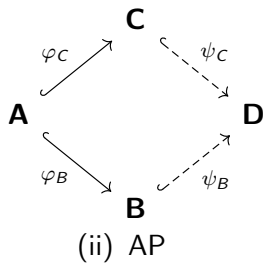
- 1  $\mathcal{V}$  has the congruence extension property.
- 2  $\mathcal{V}$  has the extension property.
- 3  $\mathcal{V}_{\text{FSI}}$  has the congruence extension property.
- 4  $\mathcal{V}_{\text{FSI}}$  has the extension property.

## Definition:

Let  $\mathcal{K}, \mathcal{K}'$  be classes of algebras in a common language.

- An **amalgam in  $\mathcal{K}'$**  of a doubly injective span  $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$  in  $\mathcal{K}$  is a triple  $\langle \mathbf{D}, \psi_B, \psi_C \rangle$  where  $\mathbf{D} \in \mathcal{K}'$  and  $\psi_B, \psi_C$  are embeddings of  $\mathbf{B}$  and  $\mathbf{C}$  into  $\mathbf{D}$ , respectively, such that  $\psi_B \varphi_B = \psi_C \varphi_C$ .
- We say  $\mathcal{K}$  has the **amalgamation property** (or AP) if every doubly injective span in  $\mathcal{K}$  has an amalgam in  $\mathcal{K}$
- We say  $\mathcal{K}$  has the **one-sided amalgamation property** (or 1AP) if for any doubly injective span  $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$  in  $\mathcal{K}$ , there exist a  $\mathbf{D} \in \mathcal{K}$ , a **homomorphism**  $\psi_B: \mathbf{B} \rightarrow \mathbf{D}$ , and an embedding  $\psi_C: \mathbf{C} \rightarrow \mathbf{D}$  such that  $\psi_B \varphi_B = \psi_C \varphi_C$ .

# The AP and 1AP



## Theorem:

Let  $\mathcal{V}$  be a variety with the congruence extension property such that  $\mathcal{V}_{\text{FSI}}$  is closed under subalgebras. The following are equivalent:

- 1  $\mathcal{V}$  has the amalgamation property.
- 2  $\mathcal{V}$  has the one-sided amalgamation property.
- 3  $\mathcal{V}_{\text{FSI}}$  has the one-sided amalgamation property.
- 4 Every doubly injective span of finitely generated algebras from  $\mathcal{V}_{\text{FSI}}$  has an amalgam in  $\mathcal{V}_{\text{FSI}} \times \mathcal{V}_{\text{FSI}}$ .
- 5 Every doubly injective span of finitely generated algebras from  $\mathcal{V}_{\text{FSI}}$  has an amalgam in  $\mathcal{V}$ .

## Definition:

A variety  $\mathcal{V}$  is **finitely generated** if it is generated by a finite set of finite algebras of finite signature. A variety is **residually small** if there is a cardinal bound on the size of its subdirectly irreducible members.

By applying Jónsson's Lemma and the results given previously, we obtain the following decidability result.

## Theorem:

Let  $\mathcal{V}$  be a finitely generated congruence-distributive variety such that  $\mathcal{V}_{\text{FSI}}$  is closed under subalgebras. There exist **effective algorithms** to decide if  $\mathcal{V}$  has the congruence extension property and amalgamation property.

# The algorithm

**Step 1:** By Jónsson's lemma, construct a finite set  $\mathcal{V}_{\text{FSI}}^* \subseteq \mathcal{V}_{\text{FSI}}$  of finite algebras such that each  $\mathbf{A} \in \mathcal{V}_{\text{FSI}}$  is isomorphic to some  $\mathbf{A}^* \in \mathcal{V}_{\text{FSI}}^*$ .

**Step 2:** Check if each member of  $\mathcal{V}_{\text{FSI}}^*$  has the congruence extension property to see if  $\mathcal{V}$  has the CEP.

**Step 3:** It is known that if a residually small, congruence-distributive variety has the AP, then it has the CEP. Since  $\mathcal{V}$  is residually small, if  $\mathcal{V}$  does not have the CEP in Step 2, then  $\mathcal{V}$  does not have the AP.

**Step 4:** If otherwise  $\mathcal{V}$  has the CEP, it can be decided if  $\mathcal{V}$  has the AP by checking if  $\mathcal{V}_{\text{FSI}}$  has the 1AP.



**Kleene lattices** are generated by a single 3-element algebra: A totally order lattice  $\{-1, 0, 1\}$  with  $-1 < 0 < 1$  with the binary operations of  $\wedge$  for minimum and  $\vee$  for maximum, and negation  $\neg$  given by the additive inversion, plus  $-1$  and  $1$  named by constant symbols. Call this algebra  $\mathbf{L}$ . The variety  $\mathcal{V}$  of Kleene lattices is the variety generated by  $\mathbf{L}$ .

**Step 1:** Up to isomorphism only two non-trivial finitely subdirectly irreducibles:  $\mathbf{L}$  and the 2-element Boolean algebra  $\mathbf{B}(= \{-1, 1\})$ , so  $\mathcal{V}_{\text{FSI}}^* = \{\mathbf{L}, \mathbf{B}\}$ .

**Step 2:** Easy to verify that both  $\mathbf{L}$  and  $\mathbf{B}$  have the CEP by directly computing congruences and subalgebras.

**Step 3:** Since  $\mathcal{V}$  has the CEP we go on to Step 4.

**Step 4:** Since  $\mathbf{L}$  does not embed in  $\mathbf{B}$ , there are just three doubly-injective spans to check, and the only non-trivial one (the one where  $\mathbf{B}$  embeds as a common subalgebra of  $\mathbf{L}$  and a  $\mathbf{B}$ ). Each of these can be completed, so  $\mathcal{V}$  has the 1AP.

We conclude that  $\mathcal{V}$  has the amalgamation property.

- We have only run through a toy example, but it reconstructs a result of Cignoli (1975) and later Cornish and Fowler (1977).
- In the paper, we apply these results to classify whether certain extensions of Hájek's basic fuzzy logic have deductive interpolation/amalgamation.
- Other applications are to extensions of the Full Lambek calculus by the contraction and mingle rules (F.-Galatos 2022+).
- Many other probable use cases (modal logics?)

Thank you!

# Thank you!

For more information, see the paper at [arXiv:2205.05148](https://arxiv.org/abs/2205.05148)