Interpolation via Finitely Subdirectly Irreducible Algebras

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Metalogical properties and algebra

This talk is about practical algebraic tools to study metalogical properties. In particular, if $\vdash$ is a deductive system we say:

- $\vdash$ has the deductive interpolation property if for all formulas φ, ψ such that $\varphi \vdash \psi$, there exists a formula σ such that $\varphi \vdash \sigma$, $\sigma \vdash \psi$, and the variables of σ are among those appearing in both of φ and ψ.
- $\vdash$ has a local deduction theorem if there exists a family $\{d_j(p,q) : j \in J\}$ of sets of formulas $d_j(p,q)$ in at most two variables such that for every set of formula $\Gamma \cup \{\varphi, \psi\}$ we have

$$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash d_j(\varphi, \psi) \text{ for some } j \in J.$$

  (Think $d_j(\varphi, \psi) = \{\varphi \to \psi\}$).

When $\vdash$ is algebraizable, these properties correspond to the the amalgamation property and the the congruence extension property.

The breadth of application

- Most everyday logics are algebraizable (see Blok-Pigozzi 1989):
  - Intuitionistic logic, normal modal logics, multiplicative-additive linear logic, relevance logics above **R**, Łukasiewicz logic, t-norm based fuzzy logics, classical first-order logic, ...
- This study applies in a language-agnostic way (may not have $\to$, can't express Craig interpolation).
  - Consider fragments of well-known logics.
- Applies even when there is no good proof theory.
  - Challenging extensions of Full Lambek calculus like GBL, Łukasiewicz, ...
- Main limitation is set by whether it is difficult to analyze the pertinent algebraic models.

Some basic definitions

- A variety is a class $\mathcal{V}$ of algebras defined by equations, or equivalently, closed under homomorphic images, direct products, and subalgebras.
- An algebra $\mathbf{A}$ is called subdirectly irreducible if whenever $\mathbf{A}$ is isomorphic to a subdirect product of a set of algebras, it is isomorphic to one of these algebras. Equivalently, $\mathbf{A}$ is subdirectly irreducible if the least congruence $\Delta_A = \{\langle a, a\rangle : a \in A\}$ is completely meet-irreducible in the lattice of congruences $\mathrm{Con}\,\mathbf{A}$ of $\mathbf{A}$. We denote the subdirectly irreducibles of a variety $\mathcal{V}$ by $\mathcal{V}_{\mathsf{SI}}$.
- An algebra $\mathbf{A}$ is called finitely subdirectly irreducible if whenever $\mathbf{A}$ is isomorphic to a subdirect product of a non-empty, finite set of algebras, it is isomorphic to one of these algebras. Equivalently, $\mathbf{A}$ is finitely subdirectly irreducible if Δ_A is meet-irreducible in $\mathrm{Con}\,\mathbf{A}$. We denote the finitely subdirectly irreducibles of a variety $\mathcal{V}$ by $\mathcal{V}_{\mathsf{FSI}}$.

From SI algebras to FSI algebras

- Establishing that a variety $\mathcal{V}$ has some property by arguing on $\mathcal{V}_{\mathsf{SI}}$ is a common proof strategy in universal algebra.
- Theme of today's talk: One often obtains simpler, more elegant, and more useful formulations of transfer theorems (from $\mathcal{V}_{\mathsf{FSI}}$ to $\mathcal{V}$) when working with FSI algebras instead of SI algebras, especially for logically-relevant properties and varieties.
- One reason: Varieties corresponding to logics often have equationally definable principal meets, which implies $\mathcal{V}_{\mathsf{FSI}}$ is a universal class.
- Often obtain equivalent formulations of properties in terms of $\mathcal{V}_{\mathsf{FSI}}$ that do not exist for $\mathcal{V}_{\mathsf{SI}}$.

The congruence extension property

Definition:

An algebra $\mathbf{A}$ is congruence-distributive if $\mathrm{Con}\,\mathbf{A}$ is a distributive lattice. $\mathbf{A}$ has the congruence extension property (or CEP) if for any subalgebra $\mathbf{A}$ of $\mathbf{B}$ and $\Theta \in \mathrm{Con}\,\mathbf{A}$, there exists a $\Phi \in \mathrm{Con}\,\mathbf{B}$ such that $\Phi \cap A^2 = \Theta$. A class of algebras $\mathcal{K}$ has these properties if each $\mathbf{A} \in \mathcal{K}$ does.

Our first illustration of FSI algebras in action:

Theorem:

Let $\mathcal{V}$ be any congruence-distributive variety. Then $\mathcal{V}$ has the congruence extension property if and only if $\mathcal{V}_{\mathsf{FSI}}$ has the congruence extension property.

Comparison to SI algebras

The previous theorem improves on the following similar results for SIs, which is also a corollary by noting that each member of $\mathcal{V}_{\mathsf{FSI}}$ embeds into an ultraproduct of members of $\mathcal{V}_{\mathsf{SI}}$:

Corollary (Davey 1977):

Let $\mathcal{V}$ be any congruence-distributive variety such that $\mathcal{V}_{\mathsf{SI}}$ is an elementary class. Then $\mathcal{V}$ has the congruence extension property if and only if $\mathcal{V}_{\mathsf{SI}}$ has the congruence extension property.

Major improvement is that $\mathcal{V}_{\mathsf{SI}}$ being an elementary class is a very strong hypothesis, and not needed for $\mathcal{V}_{\mathsf{FSI}}$.

Spans

Often the CEP is reformulated using commutative diagrams.

Definition:

Let $\mathcal{K}$ be a class of similar algebras. A span in $\mathcal{K}$ is a 5-tuple $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ consisting of $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and homomorphisms $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$. A span is:

- injective if φ_B is an embedding.
- doubly injective if both φ_B and φ_C are embeddings.
- injective-surjective if φ_B is an embedding and φ_C is surjective.

The extension property

Definition:

A class $\mathcal{K}$ of similar algebras has the extension property (or EP) if for any injective-surjective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$.

Proposition (Bacsich 1972):

A variety $\mathcal{V}$ has the CEP if and only if it has the EP.

Note: This need not be true for other classes of algebras.

The extension property

$$\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\
\mathbf{A} \arrow[ur, two heads, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, two heads, dashed, "\psi_B"'] &
\end{tikzcd}$$

(i) EP

A transfer theorem for the EP

Our previous characterization of the CEP in terms of $\mathcal{V}_{\text{FSI}}$ does not make any special demands on $\mathcal{V}_{\text{FSI}}$ itself, but with additional hypotheses we can say more:

Theorem:

Let $\mathcal{V}$ be a congruence-distributive variety such that $\mathcal{V}_{\text{FSI}}$ is closed under subalgebras. The following are equivalent:

1. $\mathcal{V}$ has the congruence extension property.
2. $\mathcal{V}$ has the extension property.
3. $\mathcal{V}_{\text{FSI}}$ has the congruence extension property.
4. $\mathcal{V}_{\text{FSI}}$ has the extension property.

The AP and 1AP

Definition:

Let $\mathcal{K}, \mathcal{K}'$ be classes of algebras in a common language.

- An amalgam in $\mathcal{K}'$ of a doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$ is a triple $\langle \mathbf{D}, \psi_B, \psi_C \rangle$ where $\mathbf{D} \in \mathcal{K}'$ and ψ_B, ψ_C are embeddings of $\mathbf{B}$ and $\mathbf{C}$ into $\mathbf{D}$, respectively, such that $\psi_B \varphi_B = \psi_C \varphi_C$.
- We say $\mathcal{K}$ has the amalgamation property (or AP) if every doubly injective span in $\mathcal{K}$ has an amalgam in $\mathcal{K}$
- We say $\mathcal{K}$ has the one-sided amalgamation property (or 1AP) if for any doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$.

The AP and 1AP

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\
\mathbf{A} \arrow[ur, hook, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, hook, dashed, "\psi_B"'] &
\end{tikzcd}
$$

(ii) AP

$$
\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\
\mathbf{A} \arrow[ur, hook, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] &
\end{tikzcd}
$$

(iii) 1AP

A transfer theorem for the AP

Theorem:

Let $\mathcal{V}$ be a variety with the congruence extension property such that $\mathcal{V}_{\mathsf{FSI}}$ is closed under subalgebras. The following are equivalent:

1. $\mathcal{V}$ has the amalgamation property.
2. $\mathcal{V}$ has the one-sided amalgamation property.
3. $\mathcal{V}_{\mathsf{FSI}}$ has the one-sided amalgamation property.
4. Every doubly injective span of finitely generated algebras from $\mathcal{V}_{\mathsf{FSI}}$ has an amalgam in $\mathcal{V}_{\mathsf{FSI}} \times \mathcal{V}_{\mathsf{FSI}}$.
5. Every doubly injective span of finitely generated algebras from $\mathcal{V}_{\mathsf{FSI}}$ has an amalgam in $\mathcal{V}$.

Decidability

Definition:

A variety $\mathcal{V}$ is finitely generated if it is generated by a finite set of finite algebras of finite signature. A variety is residually small if there is a cardinal bound on the size of its subdirectly irreducible members.

By applying Jónsson's Lemma and the results given previously, we obtain the following decidability result.

Theorem:

Let $\mathcal{V}$ be a finitely generated congruence-distributive variety such that $\mathcal{V}_{\mathsf{FSI}}$ is closed under subalgebras. There exist effective algorithms to decide if $\mathcal{V}$ has the congruence extension property and amalgamation property.

The algorithm

Step 1: By Jónsson's lemma, construct a finite set $\mathcal{V}^*_{\mathsf{FSI}} \subseteq \mathcal{V}_{\mathsf{FSI}}$ of finite algebras such that each $\mathbf{A} \in \mathcal{V}_{\mathsf{FSI}}$ is isomorphic to some $\mathbf{A}^* \in \mathcal{V}^*_{\mathsf{FSI}}$.

Step 2: Check if each member of $\mathcal{V}^*_{\mathsf{FSI}}$ has the congruence extension property to see if $\mathcal{V}$ has the CEP.

Step 3: It is known that if a residually small, congruence-distributive variety has the AP, then it has the CEP. Since $\mathcal{V}$ is residually small, if $\mathcal{V}$ does not have the CEP in Step 2, then $\mathcal{V}$ does not have the AP.

Step 4: If otherwise $\mathcal{V}$ has the CEP, it can be decided if $\mathcal{V}$ has the AP by checking if $\mathcal{V}_{\mathsf{FSI}}$ has the 1AP.

An example

Kleene lattices are generated by a single 3-element algebra: A totally order lattice $\{-1, 0, 1\}$ with $-1 < 0 < 1$ with the binary operations of $\wedge$ for minimum and $\vee$ for maximum, and negation $\neg$ given by the additive inversion, plus -1 and 1 named by constant symbols. Call this algebra $\mathbf{L}$. The variety $\mathcal{V}$ of Kleene lattices is the variety generated by $\mathbf{L}$.

Step 1: Up to isomorphism only two non-trivial finitely subdirectly irreducibles: $\mathbf{L}$ and the 2-element Boolean algebra $\mathbf{B}(= \{-1, 1\})$, so $\mathcal{V}^*_{\mathsf{FSI}} = \{\mathbf{L}, \mathbf{B}\}$.

Step 2: Easy to verify that both $\mathbf{L}$ and $\mathbf{B}$ have the CEP by directly computing congruences and subalgebras.

An example

Step 3: Since $\mathcal{V}$ has the CEP we go on to Step 4.

Step 4: Since **L** does not embed in **B**, there are just three doubly-injective spans to check, and the only non-trivial one (the one where **B** embeds as a common subalgebra of **L** and a **B**). Each of these can be completed, so $\mathcal{V}$ has the 1AP.

We conclude that $\mathcal{V}$ has the amalgamation property.

Conclusion

- We have only run through a toy example, but it reconstructs a result of Cignoli (1975) and later Cornish and Fowler (1977).
- In the paper, we apply these results to classify whether certain extensions of Hájek's basic fuzzy logic have deductive interpolation/amalgamation.
- Other applications are to extensions of the Full Lambek calculus by the contraction and mingle rules (F.-Galatos 2022+).
- Many other probable use cases (modal logics?)

Thank you!

For more information, see the paper at arXiv:2205.05148